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THE LOGARITHMIC LAW IN TURBULENT BOUNDARY LAYERS: THE DEBATE CONTINUES

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ABSTRACT

In the present research, we analyze the experimental and DNS data sets of canonical turbulent boundary layers from six independent groups. For the range of momentum-thickness Reynolds numbers of 500–27,320, we determine the best-fit values for the parameters appearing in the standard logarithmic law and the power law. The result is that neither the standard logarithmic law nor the proposed alternative power law is valid throughout the entire overlap region.

The remedy for the problem is an extension of the logarithmic law to higher orders concerning the von Kármán number. It is found that the standard inner logarithmic law with κ=0.393 and C_{log}=4.568 is the envelope of this generalized law for an infinite von Kármán number δ^+. The appropriate outer logarithmic law has an intercept of C_I=3.912.

INTRODUCTION

There has been considerable controversy during the past few years concerning the validity of the standard logarithmic law for the overlap region of the mean velocity profile in the canonical turbulent boundary layer. Alternative power laws have been proposed by George and Castillo (1997) and Barenblatt *et al.* (2000), among others.

One way to derive the standard logarithmic law is to recognize the two-scale nature of the turbulent boundary layer. Close to a smooth wall (the inner region) viscosity is important and the proper velocity and length scales are the friction velocity u_τ and the viscous length scale (or wall unit) ν/u_τ, respectively, where ν is the kinematic viscosity and $u_\tau = \sqrt{\tau_w / \rho}$, τ_w is the shear stress at the wall and ρ is the fluid density. At a sufficient distance from the wall (the outer region) inertia is of importance and the proper velocity and length scales are the velocity defect (U_o-u) and the boundary layer thickness δ, respectively, where U_o is the velocity outside the boundary layer and u is the mean streamwise velocity. An overlap region is presumed to exist for distances from the wall $\nu/u_\tau \ll y \ll \delta$, or roughly, and empirically, from $y=30\nu/u_\tau$ to $y=0.2/\delta$. Dimensional reasoning or asymptotic analysis yields the standard logarithmic law for the mean velocity profile in the overlap region. Written in terms of the inner variables the log law reads

$$u^+ = \frac{1}{\kappa} \ln y^+ + C_{log} \tag{1}$$

where the von Kármán constant κ and the intercept C_{log} are usually assumed to be independent of the Reynolds number. The standard logarithmic law according to eq. (1) uses constant values of κ=0.40–0.41 and C_{log}=5.0–5.1. For low Re_θ these parameters are confirmed in a recent work by Osaka *et al.* (1998). According to their experimental results these parameters are κ=0.41 and C_{log}=4.9. However, using these parameters the overlap region of the mean velocity profile can be fitted succesfully only in a certain range of the momentum-thickness Reynolds numbers Re_θ. For moderate and high Re_θ (Re_θ>6,000) Österlund *et al.* (2000) therefore found κ=0.38 and C_{log}=4.1.

For alternative power laws (2) it is assumed a priori that the parameters–the power α and the coefficient C_{pow}–are Reynolds number dependent.

$$u^+ = C_{pow} \left(y^+\right)^\alpha \tag{2}$$

NOMENCLATURE

α, C_{Pow}	parameters of the power law
κ, C_{log}	parameters of the standard logarithmic law
FD	fractional difference
u^+, y^+	inner variables
U, η	outer variables
C_1	intercept of the standard outer logarithmic law
δ^+	von Kármán number

ANALYSIS OF DATA

Eight sets of velocity profiles obtained experimentally or using direct numerical simulation by six independent groups of researchers were analyzed (see table 1). The test cases include 109 velocity profiles and cover a range of momentum Reynolds number of $500<Re_\theta<27{,}320$. The shape parameters has a range of $1.292<H_{12}<1.509$.

Table 1: Analyzed sets of velocity profiles (VP) of zero pressure gradient turbulent boundary layers
(DNS - Direct numerical simulation, SHW - Single-hot-wire-probe, XW - X-wire probe)
K. S. Choi (private communications)

Author	Remarks	Symbol	Range of Re_θ
Meinert, M. (2000)	6 VP / SHW		2,442–6,167
Österlund, J. (1999)	70 VP / SHW	■	2,530–27,320
Osaka, H. *et al.* (1998)	14 VP SHW, XW	▲	860–6,040
Choi, K. S.	1 VP / SHW	★	630–1,140
Roach, P. E. & Brierley, D. H. (1990)	16 VP SHW	●	500–2,700
Spalart, P. (1989)	2 VP / DNS	◆	640; 1,410

For the determination of the parameters of the logarithmic law (1) (κ, C_{log}) and the power law (2) (α, C_{Pow}) the fractional difference FD (3) of the mean velocity profile was plotted against y^+.

$$FD\% = 100\left(1 - \frac{u_P^+}{u_{ED}^+}\right) \qquad (3)$$

Here u_P^+ is the velocity value according to eq. (1) or (2), and u_{ED}^+ is the velocity value from experiments or DNS. The parameters were varied until as many as possible FD-values were found in a region of ±0.5%. The smaller the fractional difference is, the better are the experimental data reproduced by the law applied. If one of the laws shows significantly smaller FD-values in a certain region of the profile, then this law should be preferred there.

Figure 1 shows the FD-distribution of a DNS velocity profile from Spalart (1989) and a high Reynolds number profile from Österlund (1999). The distributions clearly show that the logarithmic law and the power law do not cover the same region of the velocity profile. A very small zone directly above the buffer layer is not represented by the power law. On the other hand, the inner part of the region, usually called wake zone, is covered by it. In the region between those two zones the discrepancy between the power law and experimental values, on the one hand and the logarithmic law and experiments / DNS, on the other hand is similarly small. In this region it is not possible to decide directly from the FD-values which approach has to be preferred. This result is in agreement with the conclusions from theoretical considerations by Panton (2000).

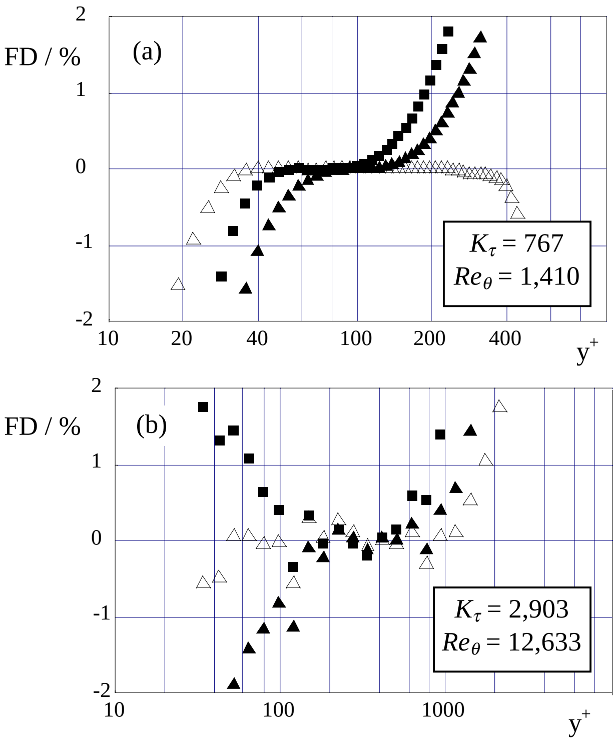

Figure 1. Fractional difference of the mean velocity profile
■ logarithmic law, ▲ power law, Δ eq. (14)
(a) Spalart's DNS data (1989) (b) Österlund (1999)

In addition it is found that both the parameters of the logarithmic law and the power law are Reynolds number dependent but to various degrees. While the power α and the coefficient C_{Pow} are Reynolds number dependent throughout the entire Reynolds number interval investigated, the parameters of the log law seem to reach constant values. The consequence of these findings is that the standard logarithmic law should be extended to higher-order terms which include Reynolds number effects.

EXTENSION OF THE LOGARITHMIC LAW

For fully developed turbulent pipe and channel flow, Afzal (1976) derived a second-order extension of the standard logarithmic law for moderately large Reynolds numbers. The basic idea proposed by Afzal will be used to derive a generalized logarithmic law.

Lets us assume a canonical turbulent boundary layer which is free of any effects concerning pressure gradients, roughness and outer turbulence. The asymptotic expansions for the inner and the outer layer are then written as

$$u^+ \sim u_1^2\left(y^+\right) + u_2^2\left(y^+\right)\gamma_2\left(K_\tau\right) + u_3^2\left(y^+\right)\gamma_3\left(K_\tau\right) + \ldots \quad (4)$$

$$U \sim U_1^2\left(y^+\right) + U_2^2\left(y^+\right)\Gamma_2\left(K_\tau\right) + U_3^2\left(y^+\right)\Gamma_3\left(K_\tau\right) + \ldots \quad (5)$$

Here δ^+ denotes the von Kármán number

$$\delta^+ = \delta\, u_\tau \,/\, \nu \quad (6)$$

In (4, 5) the von Kármán number dependences are separated into gauge functions g_i and G_i which are not necessarily identical. Already Panton (1990) stated that 'while the first term in an expansion is reasonably well defined, subsequent terms can differe depending on the sequence of gauge functions choosen'. Based on a discussion of the governing equation of mean motion of fully developed pipe and channel flow, Afzal (1976) introduced the following gauge functions for a second-order analysis :

$$\gamma_2 = \Gamma_2 = \varepsilon_2 = 1 \big/ \delta^+ \quad (7)$$

This type of gauge functions can be generalized as

$$\gamma_i = \Gamma_i = \varepsilon_i = 1 \big/ \delta^{+i} \quad (8)$$

The split of the mean velocity profile into two expansions causes a loss of boundary conditions for each expansion. To compensate this deficit the matching condition according to Millikan (1938) has to be introduced.

$$y^+ \frac{\partial u^+}{\partial y^+} \approx \eta \frac{\partial U}{\partial \eta} \quad (9)$$

Introducing (4, 5) into (9) leads to the lowest order matching condition as $y^+ \to \infty$ and $\eta \to 0$ as $\varepsilon \to 0$

$$y^+ \frac{\partial u_1^+}{\partial y^+} \approx \eta \frac{\partial U_1}{\partial \eta} + O(\varepsilon) \quad (10)$$

Each side of (10) has to be asymptotic to a constant

$$y^+ \frac{\partial u_1^+}{\partial y^+} = \frac{1}{\kappa_1} + O\left(y^{+-t}\right) \quad (11\text{ a})$$

$$\eta \frac{\partial U_1}{\partial \eta} = \frac{1}{\kappa_1} + O\left(\eta^s\right) \quad (11\text{ b})$$

Neclecting the higher-order terms concerning y^+ and η, the integration of (11) leads to the standard logarithmic law in inner and outer variables. However, to find the higher order-terms in ε one has first to find the higher order terms of y^+ and η in this lowest order solution (cf. Gill, 1968). With respect to the next higher gauge function Afzal (1976) necessarily concluded that these terms have to be integral powers of $1/y^+$ and η. The integration of (11) than yields

$$u_1^+ = \frac{1}{\kappa_1} ln\left[y^+\right] + C_1 + \frac{E_{1,1}}{y^+} + \ldots \quad (12\text{ a})$$

$$U_1 = \frac{1}{\kappa_1} ln[\eta] + c_1 + e_{1,1}\,\eta + \ldots \quad (12\text{ b})$$

Introducing (12) into (9) and collecting the terms of order ε we obtain

$$y^+ \frac{\partial u_2^+}{\partial y^+} - e_{1,1}\, y^+ \approx \eta \frac{\partial U_2}{\partial \eta} + \frac{E_{1,1}}{\eta} + O\left(\varepsilon^2\right) \quad (13)$$

Assuming that each side of (12) is again asymptotic to a constant say, $1/\kappa_2$, for u_2^+ and U_2 follows

$$y^+ \frac{\partial u_2^+}{\partial y^+} - e_{1,1}\, y^+ = \frac{1}{\kappa_2} + O\left(y^{+-p}\right) \quad (14\text{ a})$$

$$\eta \frac{\partial U_2}{\partial \eta} + \frac{E_{1,1}}{\eta} = \frac{1}{\kappa_2} + O\left(\eta^s\right) \quad (14\text{ b})$$

After integration of (13) and obtaining a second order solution the original approach by Afzal (1976) stops here. However, the process described above can be continued so that a generalized inner logarithmic law and a generalized defect law of abitrary order are derived. In general the inner law is written as

$$u^+ = \frac{1}{\kappa_1} ln\left[y^+\right] + C_1 + \sum_{i=1}^{m} \frac{E_{1,i}}{y^{+i}} + \sum_{j=2}^{n} \varepsilon^{j-1}\left(\frac{1}{\kappa_j} ln\left[y^+\right] + C_j + \sum_{i=1}^{j-1} B_{j,i}\, y^{+i} + \sum_{i=1}^{m} \frac{E_{j,i}}{y^{+i}}\right) \quad (15)$$

and the defect law as

$$U = \frac{1}{\kappa_1} ln[\eta] + c_1 + \sum_{i=1}^{m} e_{1,i}\, \eta + \sum_{j=2}^{n} \varepsilon^{j-1} \left(\frac{1}{\kappa_j} ln[\eta] + c_j + \sum_{i=1}^{j} \frac{b_{j,i}}{\eta^i} + \sum_{i=1}^{m} e_{j,i}\, \eta \right) \qquad (16)$$

Here κ_j, C_j, c_j, $B_{j,i}$, $b_{j,i}$, $E_{j,i}$ and $e_{j,i}$ denote constants which have to be determined from experiments. Due to the matching process both laws are coupled according to the following relations

$$B_{j,i} = e_{j-i,i} \qquad (17\ a)$$

$$b_{j,i} = E_{j-i,i} \qquad (17\ b)$$

From (15, 16) it becomes clear that the usual constants of the standard logarithmic law (1) κ and C_{log} are now functions depending on the von Kármán number. Due to the asymptotic behaviour of both laws concerning δ^+ this dependence persists also for high Reynolds numbers. It has to be understood as a 'Reynolds number effect' and clearly to be distinguished from the 'low Reynolds number effects' (cf. e. g. Spalart, 1989). In general, the parameters of both laws have the form (18), where G_j can be one of the parameters from (15, 16).

$$G = G_1 + \frac{G_2}{\delta^+} + \frac{G_3}{\delta^{+2}} + \ldots = \sum_{j=1}^{n} \frac{G_j}{\delta^{+\,i-1}} \qquad (18)$$

It should be mentioned that pure logarithmic regions how they were proposed by the standard approach by Millikan (1938), do not exist from the point of view of the generalized laws. The higher-order terms of y^+ persist throughout the entire overlap zone and for all finite von Kármán numbers.

A fit of the mean velocity profiles obtained by Österlund (1999), Osaka *et al.* (1998), Roach and Brierley (1990) and Choi (1998) and the DNS-profiles by Spalart (1989) using the generalized inner logarithmic law (15) shows promising results. The parameters κ, C_{log}, B and E show the predicted dependences when applied onto δ^+ (fig. 2). The von Kármán constant (fig. 2 a), the intercept of the generalized inner law (fig. 2 b) and the intercept of the generalized defect law (fig. 4) asymptote to the values κ=0.393, C_{log}=4.568 and C_1=3.912. All other parameters asymptote to zero (see fig. 2 c, d).

To compare the standard logarithmic law with the new inner law, both are depicted in figure 3 for four different Reynolds numbers Re_θ. In all four of the test cases, a good agreement between experiment and calculation is found. Additionally it is found that a region larger than the fitting region is covered by eq. (15).

THE ENVELOPE AND THE DEPARTURE FUNCTION

To represent the entire velocity profile layer using an inner law $f(y^+)$ and a wake function $w(\pi,\eta)$ with π as the maximum deviation from the inner law was first proposed by Coles (1956).

$$u^+\left(y^+\right) = f\left(y^+\right) + \Delta u^+(\eta) = f\left(y^+\right) + \frac{\pi}{\kappa} w(\eta) \qquad (19)$$

The behaviour of the wake parameter π was often analyzed especially with respect to the question wether it is Reynolds number dependent or not. Already Coles (1956) found that π increases up to $Re_\theta\approx$6,000. However, an asymptotical state does not seem to be achieved because above $Re_\theta\approx$15,000 the wake parameter starts to decrease again very slowly (Gad-el-Hak, M. and Bandyopadhyay, P. R., 1994). Coles (1956) used an universal function for the law of the wall. Universal in this context means that this function only depends on the wall-normal coordinate y^+. For the inner law (14) discussed here such a function exists only for $\delta^+\to\infty$ $(\varepsilon\to 0)$. Therefore $f(y^+)$ has to be substituted by the envelope of the inner law.

$$u^+\left(y^+,\delta^+\right) = u_E^+\left(y^+\right) + \Delta u^+\left(\eta,\delta^+\right) \qquad (20)$$

From eq. (15) it becomes clear that the envelope u_E^+ of the inner law is identical with the first-order solution given by eq. (11 a). It should be mentioned that this envelope becomes identical with the standard logarithmic law (1) only for $y^+\to\infty$.

Rewriting and rearranging (19) allows us to calculate the departure function $\Delta u^+(\eta,\delta^+)$, while Coles' wake function is built as the deviation of an individual profile in the outer zone from its own overlap region. On the contrary, the departure function used here is the deviation of an individual profile from the envelope of the inner law. Using the data from table 1, the maximum value Π of this departure function is calculated. The obtained values are compiled in figure 5.

$$\Pi = Max\left[\Delta u^+\left(\eta,\delta^+\right)\right] = Max\left[u^+\left(y^+,\delta^+\right)_{Exp} - u_E^+\left(y^+\right)\right] \qquad (21)$$

Above $Re_\theta\approx$10,000 $(\delta^+\approx$2,000) The departure parameter Π shows a constant value of about 2.303 with 95% of all values in a band of 2.303±5%. A constant wake parameter π for the standard logarithmic law was found for very high Reynolds numbers by DeGraaff and Eaton (2000) using the constants from Österlund *et al.* (2000). This finding is in agreement with the result presented here, as the envelope of the generalized inner law becomes identical with the standard logarithmic law for $y^+\to\infty$.

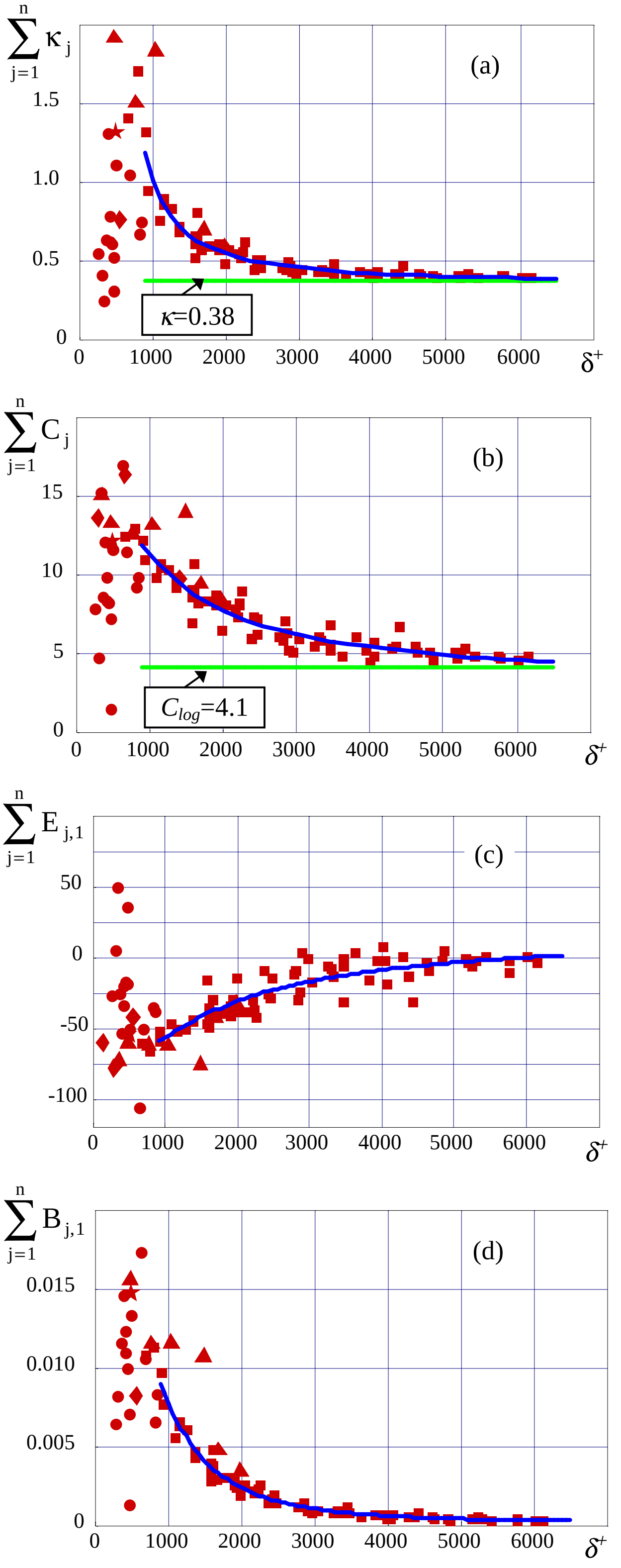

Figure 2. Universal functions of the parameters of the generalized logarithmic law
caption see figure 4

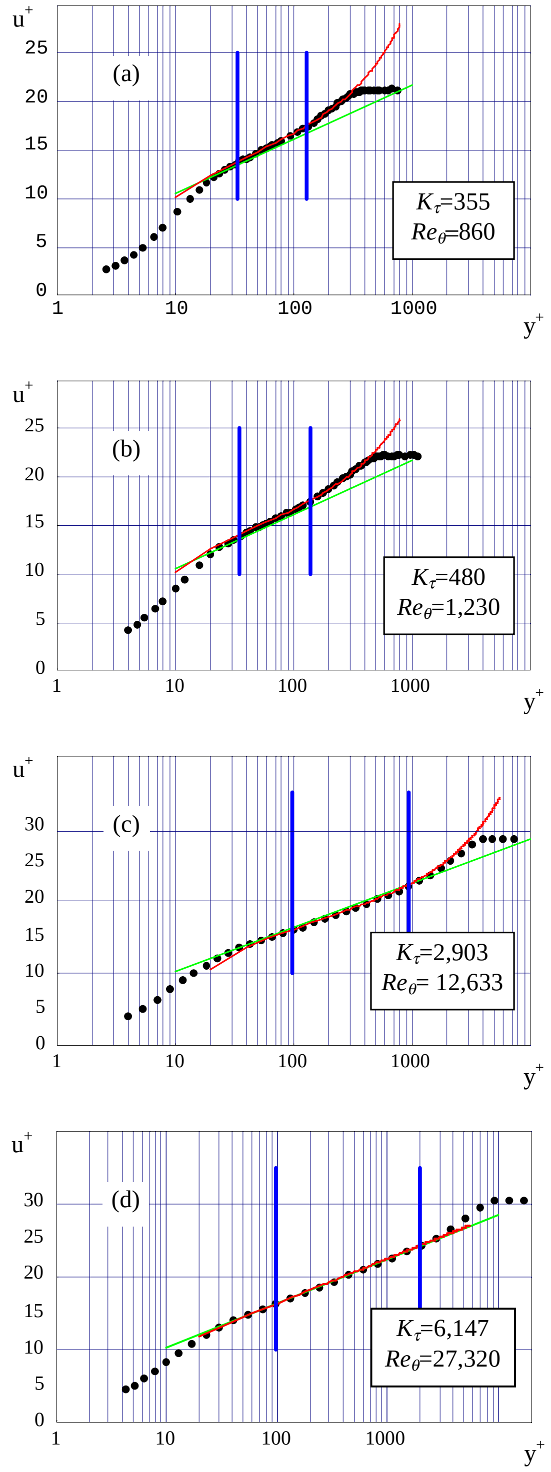

Figure 3. Experimental mean velocity profiles (black symbols) compared with the generalized logarithmic law
red : generalized inner logarithmic law
green : standard inner logarithmic law with
(a, b) κ=0.41, C_{log}=4.9 (c, d) κ=0.38, C_{log}=4.1
blue : boundaries of fit

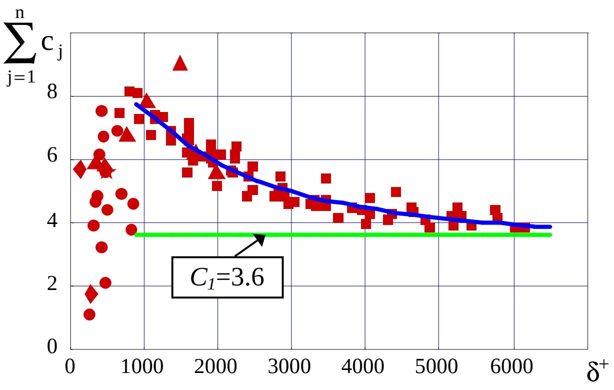

Figure 4. Universal functions of the parameters of the generalized logarithmic law (all symbols follow table 1)
blue : fit of Österlund (1999) results
green : parameters for the standard inner logarithmic law according to Österlund *et al.*(2000)

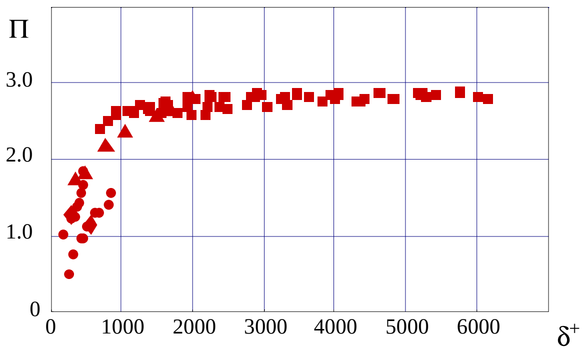

Figure 5. Maximum value of departure function Π (all symbols follow table 1)

CONCLUSION

A logarithmic law of second order which was originaly derived for pipe and channel flow is extended to higher-order terms concerning y^+ and δ^+. The resulting generalized inner logarithmic law and the generalized defect law are successfully applied to the canonical turbulent boundary layer. The parameters of these laws were derived from experiments. It is found that all parameters of these laws, including the von Kármán constant and the intercepts which are known from the standard logarithmic law asymptotically depend on the von Kármán number δ^+. All of them show the theoretically predicted dependences on δ^+. For $\delta^+ \to \infty$ the von Kármán constant κ and the intercepts C_{log} and C_1 come close to the values obtained by Österlund *et al.* (2000). All other parameters asymptote to zero. The standard logarithmic law with the Österlund-parameters is therefore the envelope of all individual Reynolds-number dependend profiles.

It is shown that the maximal departure of experimentally determined mean velocity profiles from the envelope of the generalized inner logarithmic law is constant above $\delta^+ \approx 2000$ ($Re_\theta \approx 10{,}000$).

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