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STABILITY OF A SUSPENSION IN TAYLOR COUETTE FLOW

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ABSTRACT

We apply linear hydrodynamic stability analysis to determine the critical Taylor number for the transition from stable cylindrical Couette flow to vortical flow when a dilute concentration of a discontinuous phase of rigid spherical particles is present in the fluid. The conservation equations for the system are based on the traditional two-fluid formulation. The eigenvalue problem is solved numerically to find the critical Taylor number for given particle and geometric conditions. The critical Taylor number at which Taylor vortices first appear decreases as the particle concentration increases. More dense particles have a much greater effect on the stability of the flow than less dense particles due to their inertia. Using an effective Taylor number based on the suspension density and suspension viscosity largely accounts for this effect.

INTRODUCTION

The linear stability of circular Couette flow has been studied in great detail. The instability appears as pairs of counter-rotating, toroidal vortices stacked in the annulus between a rotating inner cylinder and a fixed outer cylinder. Our interest in the effect of a a dilute suspension on the stability of Taylor Couette flow is motivated by the processing of a suspension in a Taylor-Couette reactor cell [1,2] or in a rotating filter device during dynamic filtration [3-5]. In Taylor-Couette reactors, chemically reacting species are dispersed or exposed uniformly to chemical catalysts by the vortical motion. In rotating filter devices, a suspension in the annulus between a rotating porous inner cylinder and a stationary non-porous outer cylinder is separated so that filtrate passes through the porous wall of the inner cylinder, while the concentrate is retained in the annulus.

Nearly all research on the stability of cylindrical Couette flow has been done for single-phase fluids. Very little work has focused on the effect of a suspension of particles in a fluid on the stability of the flow with some exceptions that are not directly pertinent to this study [6,7]. In this study we apply linear hydrodynamic stability analysis to determine the critical Taylor number for the transition from stable cylindrical Couette flow to vortical flow when a dilute concentration of a discontinuous phase of rigid spherical particles is present in the fluid. While the stability of the flow of a suspension in cylindrical Couette flow is a much simpler problem than that in a Taylor Couette reactor cell or in a rotating filtration device, our intent is to provide insight into the stability of the flow of a suspension in these devices.

NOMENCLATURE

A	concentration of particles (undisturbed)
d	gap between cylinders, r_2-r_1
F	interfacial interaction force
k	dimensionless wavenumber of disturbance
n	number density of particles
p	pressure
P	pressure (undisturbed)
r	radius of cylinder

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t	time
Ta	Taylor number, $\Omega_1 r_1 d/\nu$
$\bar{V}$	stable azimuthal velocity
V^*	fluid velocity
α	volume fraction
ϕ	diameter of a particle
ε	density ratio, ρ_p/ρ_f
η	radius ratio, r_1/r_2
μ	viscosity of fluid
ρ	density
σ	complex nondimensional amplification factor
Ω	rotational speed of cylinder

Subscripts

f	fluid
p	particle
r	radial direction
z	axial direction
θ	circumferential direction
1	inner cylinder
2	outer cylinder

Superscripts

*	dimensional variables
' (prime)	perturbation components

ANALYTICAL FORMULATION

The conservation equations for the system are based on the traditional two-fluid formulation for a dilute concentration of monodisperse rigid particles [8]. In this formulation, concentration-weighted forms of the continuity and Navier-Stokes equations in cylindrical coordinates (r,θ, z) are used for the continuous fluid phase and the disperse particle phase. The flow is assumed to be axisymmetric, steady, and incompressible. The cylinders are assumed to be infinitely long with the inner cylinder rotating and the outer cylinder fixed. In dimensional form, the continuity and Navier-Stokes equations for the fluid phase are

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* V_{fr}^*\right)+\frac{\partial V_{fz}^*}{\partial z^*}=0 \qquad (1a)$$

$$\frac{\partial V_{fr}^*}{\partial t^*}+V_{fr}^*\frac{\partial V_{fr}^*}{\partial r^*}-\frac{V_{f\theta}^{*\,2}}{r^*}+V_{fz}^*\frac{\partial V_{fr}^*}{\partial z^*}= \nu\left[\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* V_{fr}^*\right)\right)+\frac{\partial^2 V_{fr}^*}{\partial z^{*2}}\right]-\frac{1}{\rho_f}\frac{\partial p^*}{\partial r^*}-\frac{nF_r^*}{\rho_f} \qquad (1b)$$

$$\frac{\partial V_{f\theta}^*}{\partial t^*}+V_{fr}^*\frac{\partial V_{f\theta}^*}{\partial r^*}+\frac{V_{fr}^* V_{f\theta}^*}{r^*}+V_{fz}^*\frac{\partial V_{f\theta}^*}{\partial z^*}= \nu\left[\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* V_{f\theta}^*\right)\right)+\frac{\partial^2 V_{f\theta}^*}{\partial z^{*2}}\right]-\frac{nF_\theta^*}{\rho_f} \qquad (1c)$$

$$\frac{\partial V_{fz}^*}{\partial t^*}+V_{fr}^*\frac{\partial V_{fz}^*}{\partial r^*}+V_{fz}^*\frac{\partial V_{fz}^*}{\partial z^*}= \nu\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial V_{fz}^*}{\partial r^*}\right)+\frac{\partial^2 V_{fz}^*}{\partial z^{*2}}\right]-\frac{1}{\rho_f}\frac{\partial p^*}{\partial z^*}-g-\frac{nF_z^*}{\rho_f} \qquad (1d)$$

The equations for the particulate phase take on a similar form except that the viscous terms are absent.

$$\frac{\partial \alpha_p}{\partial t^*}+\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\alpha_p r^* V_{pr}^*\right)+\frac{\partial\left(\alpha_p V_{pz}^*\right)}{\partial z^*}=0 \qquad (2a)$$

$$\frac{\partial V_{pr}^*}{\partial t^*}+V_{pr}^*\frac{\partial V_{pr}^*}{\partial r^*}-\frac{V_{p\theta}^{*\,2}}{r^*}+V_{pz}^*\frac{\partial V_{pr}^*}{\partial z^*}= -\frac{1}{\rho_p}\frac{\partial p^*}{\partial r^*}+\frac{nF_r^*}{\alpha_p\rho_p} \qquad (2b)$$

$$\frac{\partial V_{p\theta}^*}{\partial t^*}+V_{pr}^*\frac{\partial V_{p\theta}^*}{\partial r^*}+\frac{V_{pr}^* V_{p\theta}^*}{r^*}+V_{pz}^*\frac{\partial V_{p\theta}^*}{\partial z^*}=\frac{nF_\theta^*}{\alpha_p\rho_p} \qquad (2c)$$

$$\frac{\partial V_{pz}^*}{\partial t^*}+V_{pr}^*\frac{\partial V_{pz}^*}{\partial r^*}+V_{pz}^*\frac{\partial V_{pz}^*}{\partial z^*}=-\frac{1}{\rho_p}\frac{\partial p^*}{\partial z^*}-g+\frac{nF_z^*}{\alpha_p\rho_p} \qquad (2d)$$

The volume fractions satisfy $\alpha_f+\alpha_p=1$. We consider very dilute suspensions, so α_f is approximately equal to unity. The volume fraction of particles can be related to the number density by

$$\alpha_p=\frac{n\pi\phi^{*3}}{6}$$

The interfacial force term on the right side of the momentum equations for both phases which includes the Stokes drag and added mass, is expressed as

$$F_i^*=3\pi\mu\phi^*\left(V_{fi}^*-V_{pi}^*\right)+\frac{\rho_f\pi\left(\phi^*\right)^3}{12}\frac{\partial}{\partial t^*}\left(V_{fi}^*-V_{pi}^*\right) \quad \text{for} \quad i=r,\theta,z \qquad (3)$$

[9] The drag force and added mass couple the fluid phase and particle phase equations.

The equations are nondimensionalized using d as the length scale, $r_1\Omega_1$ as the velocity scale, $\rho_f r_1^2\Omega_1^2$ as the pressure scale, and d^2/ν as the time scale. Upon nondimensionalizing the parameters that appear are the density ratio, ε, and the Taylor number, Ta, also known as the rotating Reynolds number. The nondimensional equations are similar to those used by Dimas and Kiger to analyze the linear stability of a particle-laden mixing layer [10]. Implicit in this derivation are the following assumptions: 1) The scale of the particle motion relative to the fluid motion is small. 2) The particle Reynolds number is always less than unity, so that the drag force on the particle is described by Stokes law. 3) The particles are spherical and

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rigid. 4) The suspension is sufficiently dilute to prevent hydrodynamic interactions between particles. 5) The Faxen correction to the Stokes drag is negligible. 6) The Bassett history force is negligible in comparison to the Stokes drag.

A linear stability analysis is performed by separating the variables into mean and perturbation components such that

$$V_{fr} = u'(r,z,t), \quad V_{f\theta} = \overline{V}(r) + v'(r,z,t), \quad V_{fz} = w'(r,z,t)$$
$$V_{pr} = u'_p(r,z,t), \quad V_{p\theta} = \overline{V}(r) + v'_p(r,z,t), \quad V_{pz} = w'_p(r,z,t)$$
$$p = P + p'(r,z,t), \quad \alpha_p = A + a'_p(r,z,t) \qquad (5)$$

Here, the stable velocity profile is given by $\overline{V}(r) = C_1 r + \frac{C_2}{r}$, where the constants C_1 and C_2 are functions of the radius ratio, $\eta = r_1/r_2$. The perturbations are expressed as normal modes of the form

$$\frac{u'}{u(r)} = \frac{v'}{v(r)} = \frac{w'}{w(r)} = e^{(ikz+\sigma t)}$$
$$\frac{u'_p}{u_p(r)} = \frac{v'_p}{v_p(r)} = \frac{w'_p}{w_p(r)} = e^{(ikz+\sigma t)} \qquad (6)$$
$$\frac{p'}{\omega(r)} = e^{(ikz+\sigma t)}, \quad \frac{a'_p}{a_p(r)} = e^{(ikz+\sigma t)}$$

Equations (5) and (6) are then substituted into the nondimensionalized governing equations and the stable flow terms subtracted out. After linearization the final form of the disturbance equations for the fluid phase are

$$D_* u(r) = -ikw(r) \qquad (7a)$$

$$\left(DD_* - k^2 - \sigma - \left(\frac{18A}{\phi^2} + \frac{A\sigma}{2}\right)\right)u(r) + 2Ta\frac{\overline{V}}{r}v(r) = Ta(D\omega(r)) - \left(\frac{18A}{\phi^2} + \frac{A\sigma}{2}\right)u_p(r) \qquad (7b)$$

$$\left(DD_* - k^2 - \sigma - \left(\frac{18A}{\phi^2} + \frac{A\sigma}{2}\right)\right)v(r) - Ta\left(D_*\overline{V}\right)u(r) = -\left(\frac{18A}{\phi^2} + \frac{A\sigma}{2}\right)v_p(r) \qquad (7c)$$

$$\left(DD_* - k^2 - \sigma + \frac{1}{r^2} - \left(\frac{18A}{\phi^2} + \frac{A\sigma}{2}\right)\right)w(r) = ikTa\omega(r) - \left(\frac{18A}{\phi^2} + \frac{A\sigma}{2}\right)w_p(r) \qquad (7d)$$

where the differential operators D and D_* are defined as

$$D(\ ) = \frac{d}{dr}(\ ), \quad D_*(\ ) = \frac{d}{dr}(\ ) + \frac{1}{r}(\ )$$

For the case of zero concentration (A=0), these equations reduce to those for Taylor Couette flow of a simple fluid [11].

A similar treatment of the particulate phase equations results in expressions for the particle velocity components

$$u_p(r) = \left\{\frac{2Ta\overline{V}P_o}{r\left((\sigma + P_o)^2 + 2Ta^2\left(\left(\overline{V}/r\right)D_*\overline{V}\right)\right)}\right\}v(r) + \left\{\frac{P_o(\sigma + P_o)}{\left((\sigma + P_o)^2 + 2Ta^2\left(\left(\overline{V}/r\right)D_*\overline{V}\right)\right)}\right\}u(r) - \left\{\frac{Ta(\sigma + P_o)}{\varepsilon\left((\sigma + P_o)^2 + 2Ta^2\left(\left(\overline{V}/r\right)D_*\overline{V}\right)\right)}\right\}D\omega(r) \qquad (8a)$$

$$v_p(r) = -\left\{\frac{TaP_oD_*\overline{V}}{\left((\sigma + P_o)^2 + 2Ta^2\left(\left(\overline{V}/r\right)D_*\overline{V}\right)\right)}\right\}u(r) + \left\{\frac{P_o(\sigma + P_o)}{\left((\sigma + P_o)^2 + 2Ta^2\left(\left(\overline{V}/r\right)D_*\overline{V}\right)\right)}\right\}v(r) + \left\{\frac{Ta^2D_*\overline{V}}{\varepsilon\left((\sigma + P_o)^2 + 2Ta^2\left(\left(\overline{V}/r\right)D_*\overline{V}\right)\right)}\right\}D\omega(r) \qquad (8b)$$

$$w_p(r) = \frac{P_o}{(\sigma + P_o)}w(r) - \frac{ikTa}{\varepsilon(\sigma + P_o)}\omega(r) \qquad (8c)$$

where

$$P_0 = \frac{18}{\phi^2\varepsilon} + \frac{\sigma}{2\varepsilon}$$

The particle velocity components (8) are then substituted into (7). This system of equations is solved using the boundary-value problem software package SUPORT[12] in combination with the nonlinear equation solver SNSQE [13,14]. The eigenvalue problem may be written in the implicit functional form

$$F(Ta, A, \phi, k, \varepsilon, \eta, \sigma) = 0 \qquad (9)$$

The parameters A, ϕ, k, ε, and η are usually fixed and solution of the ordinary differential equations is obtained by iteration on the eigenvalue pair (Ta, σ_i) [15,16]. Since, we seek to find the neutral stability conditions, σ_r is set to zero. At fixed radius ratio η, a search is conducted over all wave numbers k to find the minimum Taylor number, denoted as Ta_c. Our results for a single-phase fluid (A=0) are nearly identical to previously published results [17].

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RESULTS AND DISCUSSION

The critical Taylor number for transition from stable cylindrical Couette flow to supercritical Taylor vortex flow for a suspension of neutrally buoyant particles is shown in Fig. 1 as a function of the particle concentration for radius ratios ranging from η=0.45 to η=0.95. As the particle concentration increases, the flow becomes less stable. The critical wavenumber is not altered by the concentration of particles.

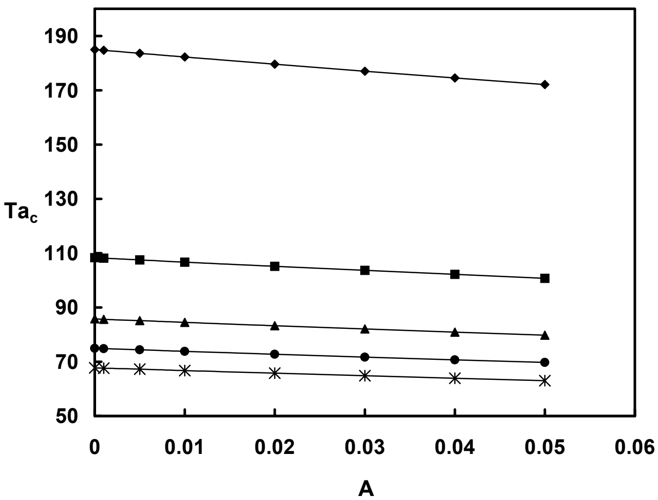

Fig 1. Critical Taylor number for several radius ratios as a function of particle concentration for neutrally buoyant particles (ε =1, ϕ=0.004). * η=0.45, ● η=0.65, ▲ η=0.75, ▪ η=0.85, ◆ η=0.95.

The effect of the particle density on the critical Taylor number is shown in Fig. 2 for three radius ratios. Here we consider particle density ratios consistent with gas bubbles in a liquid (ε =0.001), neutrally buoyant particles (ε =1), heavy particles in a liquid (ε =10), and particles about the density of water in a gas (ε =833). It is apparent that the more dense particles have a much greater effect on the stability of the flow than less dense particles. The greater effect of heavy particles on the stability may be attributed to their inertia, which results in an increased degree of coupling between the fluid and particle phases.

The presence of particles alters the effective density and viscosity of the suspension suggesting the use of an effective Taylor number based on the bulk effective density and viscosity of the suspension. The effective density is readily calculated based on the particle concentration and the density ratio. The effective viscosity is based on the Einstein formulation such that $\mu_{eff}=\mu\,(1+2.5A)$.

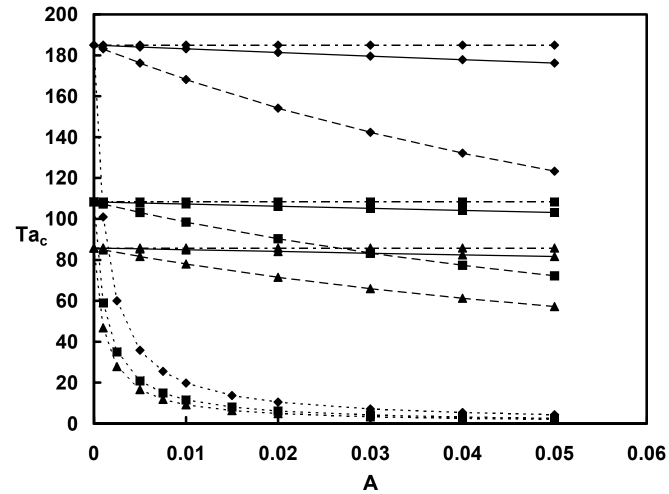

Fig 2. The effect of density ratio ε on critical Taylor number ___ _ ___ _ ε = 0.001, _____ ε = 1, __ __ __ ε = 10, - - - -ε = 833; ▲ η=0.75, ▪ η=0.85, ◆ η=0.95.

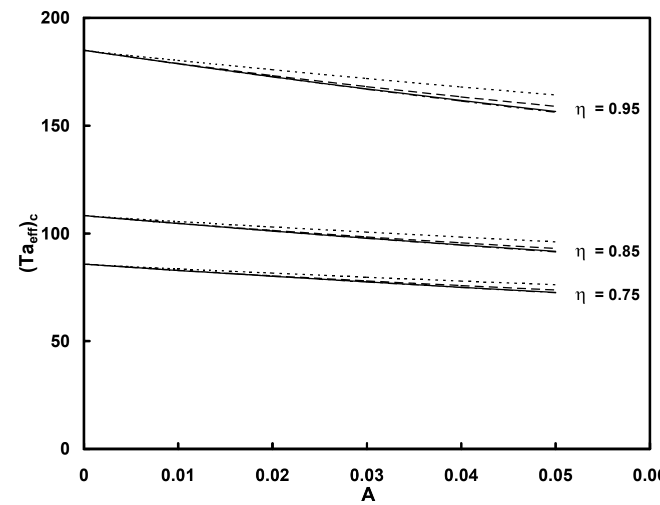

Fig 3. The effect of particle concentration on the effective critical Taylor number ___ _ ___ _ ε = 0.001 (nearly hidden by the solid line), _____ ε = 1, __ __ __ ε = 10, - - - -ε = 833.

Figure 3 indicates the dependence of the effective critical Taylor number on the concentration of particles for density ratios from ε =0.001 to ε =833 and the three radius ratios that were shown in Fig. 2. Using the effective critical Taylor number nearly collapses the data for this wide range of density ratios at all three radius ratios. Apparently, the effective suspension density, which plays a much greater role than the effective viscosity in the effective Taylor number, accounts for

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the degree to which particles follow the fluid flow and thereby affect the stability.

SUMMARY

The results of this theoretical analysis of the stability of a suspension in cylindrical Couette flow indicate that the flow is destabilized by the presence of a dispersed species. The degree of destabilization depends on the density ratio between the dispersed phase and the continuous phase. More dense particles result in more of a destabilizing effect. Most likely more dense particles (such as solid particles in a gas) are less likely to follow the fluid motion because of their inertia. This results in a stronger interaction between the disperse and continuous phases through the Stokes drag term (since the velocity difference is greater). The effect of the density ratio can be readily taken into account by using an effective Taylor number based on the bulk suspension density and viscosity.

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