

PREDICTION AND ANALYSIS OF TIDES AND TIDAL CURRENTS

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ABSTRACT

An efficient algorithm of tidal harmonic analysis and prediction is presented in this paper. Some conditions are found by means of the known approximate relationships between the harmonic constants of the tidal constituents. A system of linear equations for least squares solutions under these restricted conditions is obtained. In the case of inadequate data, ill conditioning in the system of equations that has appeared in other algorithms is conveniently avoided. In solving the resultant normal equations, the Goertzel iteration is adopted so that the whole computation time is dramatically reduced.

KEY WORDS: Tides, tidal currents, tidal harmonic analysis

INTRODUCTION

Tides and tidal currents offer clean and inexhaustible energy sources. Better prediction and analysis of tides and tidal currents are crucial to utilize hydro-dams more efficiently as energy generators. Tides are cyclic variations in the level of seas and oceans, while tidal currents are cyclic variations in the motion of seas and oceans. The present understanding of tides and tidal currents as natural phenomena due to the gravitational forces of the sun and moon acting on a rotating earth came from the development of Newton's gravitation theory [1]. Harmonic techniques were first used to analysis and predict tides and tidal currents by Thomson [2] and expanded by Darwin [3], Harris [4] and Doodson [5]. Tides and tidal currents may be considered as the sum of tidal constituents according to harmonic analysis. With the development of digital computers least squares technique is used to evaluate the tidal constituents from observed data and this is a principal method used today.

The harmonic method of tidal analysis has been further refined for improvement in accuracy of tidal prediction. A

method for superfine resolution of tidal harmonic constituents has been developed by Amin [6-8] adding a corrective step into the harmonic method. The species concordance method has been developed by George & Simon [9] and Simon [10] using relationships between species of the tide at the studied station and at a reference station where the tide is well known or easily predicted. Here we reexamine the harmonic method from a practical point of view and propose an efficient algorithm of tidal harmonic analysis and prediction.

HARMONIC METHOD FOR REGULAR OBSERVATIONS

Let us consider real-time regular observed data of tidal height $h_n = h(t_0 + n\Delta T)$ ($n = 0, 1, \dots, 2N$), where t_0 is the initial time, $2N + 1$ is the number of the real-time observed data, and ΔT is the sampling time interval. The tidal height can be expressed as a sum of cosine functions plus random errors denoted by $r_n = r(t_0 + n\Delta T)$.

$$h_n = x_0 + \sum_{m=1}^M R_m \cos[\sigma_m(t_0 + n\Delta T) - \varphi_m] + r_n, \quad n = 0, 1, \dots, 2N \quad (1)$$

where

$$R_m = f_m H_m, \quad \varphi_m = g_m - \chi_m, \quad m = 1, 2, \dots, M.$$

f_m , H_m , σ_m , g_m , and χ_m are the node factor, mean amplitude, angular velocity, epoch, and astronomical argument of the m th tidal constituent respectively. M is the number of tidal constituents.

Eq. (1) can be rewritten as

$$h_n = x_0 + \sum_{m=1}^M \{x_m \cos[(n-N)\sigma_m \Delta T] + y_m \sin[(n-N)\sigma_m \Delta T]\} + r_n, \quad (2)$$

$$n = 0, 1, \dots, 2N,$$

where

$$\{x_m, y_m\} = R_m \{\cos[\varphi_m - \sigma_m(t_0 + N \Delta T)], \sin[\varphi_m - \sigma_m(t_0 + N \Delta T)]\}$$

$$m = 1, 2, \dots, M.$$

Letting $\sigma_0 = 0$ and using matrix notation, Eq. (2) can be expressed as

$$R = H - QZ \quad (3)$$

where

$$R = \left(\frac{r_0}{\sqrt{2}}, r_1, \dots, r_{2N-1}, \frac{r_{2N}}{\sqrt{2}} \right)^T, \quad H = \left(\frac{h_0}{\sqrt{2}}, h_1, \dots, h_{2N-1}, \frac{h_{2N}}{\sqrt{2}} \right)^T.$$

$$Z^T = (X^T, Y^T),$$

and

$$X^T = (x_0, x_1, \dots, x_M), \quad Y^T = (y_1, y_2, \dots, y_M).$$

The column vectors of the coefficient matrix Q are

$$Q_0, Q_1, \dots, Q_{2M},$$

where

$$Q_m^T = \left(\frac{1}{\sqrt{2}} \cos[-N\sigma_m \Delta T], \cos[-(N-1)\sigma_m \Delta T], \dots, \right.$$

$$\left. \cos[(N-1)\sigma_m \Delta T], \frac{1}{\sqrt{2}} \cos[N\sigma_m \Delta T] \right) \quad m = 0, 1, \dots, M,$$

$$Q_m^T = \left(\frac{1}{\sqrt{2}} \sin[-N\sigma_m \Delta T], \sin[-(N-1)\sigma_m \Delta T], \dots, \right.$$

$$\left. \sin[(N-1)\sigma_m \Delta T], \frac{1}{\sqrt{2}} \sin[N\sigma_m \Delta T] \right), \quad m = M+1, M+2, \dots, 2M.$$

We make

$$\|R\|_2^2 = \|H - QZ\|_2^2 = \min,$$

where $\|R\|_2$ is an Euclidean norm of R . Then

$$\frac{\partial \|R\|_2^2}{\partial x_0} = \frac{\partial \|R\|_2^2}{\partial x_m} = \frac{\partial \|R\|_2^2}{\partial y_m} = 0, \quad m = 1, 2, \dots, M.$$

That is

$$Q^T Q Z = Q^T H. \quad (4)$$

We arrive at

$$Q^T Q = \begin{bmatrix} F & O \\ O & G \end{bmatrix}, \quad Q^T H = \begin{bmatrix} C \\ S \end{bmatrix},$$

where

$$F_{lm} = \frac{1}{2} \left\{ \frac{\sin[N(\sigma_l - \sigma_m)\Delta T]}{\tan\left[\frac{1}{2}(\sigma_l - \sigma_m)\Delta T\right]} + \frac{\sin[N(\sigma_l + \sigma_m)\Delta T]}{\tan\left[\frac{1}{2}(\sigma_l + \sigma_m)\Delta T\right]} \right\}, \quad l, m = 0, 1, \dots, M,$$

$$G_{lm} = \frac{1}{2} \left\{ \frac{\sin[N(\sigma_l - \sigma_m)\Delta T]}{\tan\left[\frac{1}{2}(\sigma_l - \sigma_m)\Delta T\right]} - \frac{\sin[N(\sigma_l + \sigma_m)\Delta T]}{\tan\left[\frac{1}{2}(\sigma_l + \sigma_m)\Delta T\right]} \right\}, \quad l, m = 1, 2, \dots, M.$$

$$C_l = \frac{1}{2} \sum_{n=1}^{2N} \{h_{n-1} \cos[(n-1-N)\sigma_l \Delta T] + h_n \cos[(n-N)\sigma_l \Delta T]\}, \quad (5)$$

$$l = 0, 1, \dots, M,$$

$$S_l = \frac{1}{2} \sum_{n=1}^{2N} \{h_{n-1} \sin[(n-1-N)\sigma_l \Delta T] + h_n \sin[(n-N)\sigma_l \Delta T]\}, \quad (6)$$

$$l = 1, 2, \dots, M.$$

Therefore Eq. (4) can be decomposed into two linear equations

$$FX = C, \quad GY = S. \quad (7)$$

The accuracy of tidal prediction can be improved as longer data time series are analyzed and more tidal constituents are selected in Eq. (7).

ITERATING ALGORITHM FOR COMPUTING C AND S

In terms of complex form

$$C_l + iS_l = \frac{1}{2} \sum_{n=1}^{2N} \{h_{n-1} e^{i(n-1-N)\sigma_l \Delta T} + h_n e^{i(n-N)\sigma_l \Delta T}\} \quad l = 1, 2, \dots, M. \quad (8)$$

Using Goertzel iteration [11],

$$\begin{cases} \rho_k = h_k + 2\rho_{k+1} \cos(\sigma_l \Delta T) - \rho_{k+2} & k = 2N-1, 2N-2, \dots, 1 \\ \rho_0 = \frac{h_0}{2} + 2\rho_1 \cos(\sigma_l \Delta T) - \rho_2, \end{cases} \quad (9)$$

under initial conditions

$$\rho_{2N+1} = 0, \rho_{2N} = \frac{h_{2N}}{2}.$$

After $2N$ time iterations, whence

$$C_l + iS_l = \left(\rho_0 - \rho_1 e^{-i\sigma_l \Delta T} \right) e^{-iN\sigma_l \Delta T}. \quad (10)$$

In this method, only $2N$ multiplications are needed.

HARMONIC METHOD FOR SEGMENTS OF REGULAR OBSERVATIONS

For K equal-length segments of observed data (overlapping is allowed). $h_n^{(k)}$ ($n=0,1,\dots,2N$, $k=1,2,\dots,K$) are observations starting at time $t_0^{(k)}$ in k th segment. The corresponding random errors are $r_n^{(k)}$. For each segment, the same M is chosen. The corresponding R and H have

$$\bar{R}_k = \left(\frac{r_0^{(k)}}{\sqrt{2}}, r_1^{(k)}, \dots, r_{N-1}^{(k)}, \frac{r_N^{(k)}}{\sqrt{2}} \right)^T, \bar{H}_k = \left(\frac{h_0^{(k)}}{\sqrt{2}}, h_1^{(k)}, \dots, h_{N-1}^{(k)}, \frac{h_N^{(k)}}{\sqrt{2}} \right)^T$$

$k = 1, 2, \dots, K.$

Then

$$\bar{R} = \bar{H} - \bar{Q}W\bar{Z}, \quad (11)$$

where

$$\bar{R}^T = (\bar{R}_1^T, \bar{R}_2^T, \dots, \bar{R}_K^T), \quad \bar{H}^T = (\bar{H}_1^T, \bar{H}_2^T, \dots, \bar{H}_K^T).$$

For $\bar{Z}^T = (\bar{X}^T, \bar{Y}^T)$, the components of \bar{X} and \bar{Y} are

$$\bar{x}_0^{(k)} = x_0^{(k)},$$

$$\left\{ \bar{x}_m^{(k)}, \bar{y}_m^{(k)} \right\} = R_m^{(k)} \left\{ \cos\left(\varphi_m^{(k)} - N\sigma_m \Delta T\right), \sin\left(\varphi_m^{(k)} - N\sigma_m \Delta T\right) \right\}, \quad (12)$$

$m = 1, 2, \dots, M.$

$$\bar{Q} = \begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & Q \end{bmatrix}$$

is a $[(2N+1)K] \times [(2M+1)K]$ matrix.

$$W^T = [W_1, W_2, \dots, W_K], \quad (13)$$

where

$$W_k = \begin{bmatrix} U_1^{(k)} & U_3^{(k)T} \\ -U_3^{(k)} & U_2^{(k)} \end{bmatrix}, \quad k = 1, 2, \dots, K,$$

and

$$U_1^{(k)} = \text{diag}\left[1, \cos(t_0^{(k)}\sigma_1), \dots, \cos(t_0^{(k)}\sigma_M)\right],$$

$$U_2^{(k)} = \text{diag}\left[\cos(t_0^{(k)}\sigma_1), \cos(t_0^{(k)}\sigma_2), \dots, \cos(t_0^{(k)}\sigma_M)\right],$$

$$U_3^{(k)} = \begin{bmatrix} 0 & \sin(t_0^{(k)}\sigma_1) & & & \\ 0 & & \sin(t_0^{(k)}\sigma_2) & & \\ \vdots & & & \ddots & \\ 0 & & & & \sin(t_0^{(k)}\sigma_M) \end{bmatrix}.$$

From

$$\|\bar{R}\|_2^2 = \|\bar{H} - \bar{Q}W\bar{Z}\|_2^2 = \min, \quad (14)$$

we have the normal equations

$$A\bar{Z} = P, \quad (15)$$

where

$$A = \sum_{k=1}^K \begin{bmatrix} U_1^{(k)}FU_1^{(k)} + U_3^{(k)T}GU_3^{(k)} & U_1^{(k)}FU_3^{(k)T} - U_3^{(k)T}GU_2^{(k)} \\ U_3^{(k)}FU_1^{(k)} - U_2^{(k)}GU_3^{(k)} & U_3^{(k)}FU_3^{(k)T} + U_2^{(k)}GU_2^{(k)} \end{bmatrix},$$

and $P^T = (\bar{C}^T, \bar{S}^T)$. The components of \bar{C} and \bar{S} are

$$\bar{C}_l = \frac{1}{2} \sum_{k=1}^K \sum_{n=1}^{2N} \left\{ h_{n-1}^{(k)} \cos\left[\left(n-1-N + \frac{t_0^{(k)}}{\Delta T}\right)\sigma_l \Delta T\right] + h_n^{(k)} \cos\left[\left(n-N + \frac{t_0^{(k)}}{\Delta T}\right)\sigma_l \Delta T\right] \right\} \quad l = 0, 1, \dots, M,$$

$$\bar{S}_l = \frac{1}{2} \sum_{k=1}^K \sum_{n=1}^{2N} \left\{ h_{n-1}^{(k)} \sin\left[\left(n-1-N + \frac{t_0^{(k)}}{\Delta T}\right)\sigma_l \Delta T\right] + h_n^{(k)} \sin\left[\left(n-N + \frac{t_0^{(k)}}{\Delta T}\right)\sigma_l \Delta T\right] \right\} \quad l = 1, 2, \dots, M.$$

CONSTRAINED CONDITIONS FOR INADEQUATE OBSERVED DATA

In the circumstances of analyzed data with insufficient-length (manly tidal currents), the tidal constituents can not be

If the above algorithm is used to analyze the tidal constituents, the total analyzed data must have sufficient length. Otherwise ill conditioning in the system of equations appears so that conventional algorithm can not separate tidal constituents effectively. Consequently in the circumstances of insufficient data (manly tidal currents), some constrained conditions can be established based on known approximate relationships among the harmonic constants of the tidal constituents. Then the lease squares solutions can be obtained from these constrained conditions. To various circumstances, the resultant linear equations can be deduced from this algorithm in order to avoid appropriately the emergence of ill conditioning. Because the constrained conditions can reasonably be established, usually final solutions do not have to be corrected so that the harmonic constants can be determined with sufficient accuracy.

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