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SOLUTION OF INCOMPRESSIBLE EULER EQUATION BY AN OPTIMALLY EFFICIENT MULTIGRID TECHNIQUE

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ABSTRACT

A fast multigrid solver for the steady incompressible inviscid Euler equations in 2D is presented. A projection operator is applied to the system of equations to factorize it into elliptic (Poisson equation for the pressure) and hyperbolic (advection) subsystems. Because the elliptic and advection parts of the system are decoupled, ideal multigrid efficiency can be achieved. The Poisson equation for the pressure may be treated by multigrid, while the advection terms of the momentum equations are treated by space marching. Using Gauss-Seidel relaxation ordered in the flow direction, textbook multigrid convergence rates are achieved.

This paper presents a first order finite difference scheme for the Euler equations and the boundary conditions. The details of implementing the multigrid V-cycle are given and numerical examples are presented. The accuracy has been demonstrated by comparison with exact solutions.

INTRODUCTION

Full multigrid (FMG) algorithms (1-3) are among the fastest solvers for elliptic problems. These algorithms can solve a general discretized elliptic problem to the discretization accuracy in a computational work that is a small (less than 10) multiple of the operation count required to perform one relaxation on the computational grid. Such efficiency is known as textbook multigrid efficiency (TME) (4-6). Extending TME to solutions of the Navier–Stokes equations is a challenging task because the Navier–Stokes equations form a system of coupled nonlinear equations that is not fully elliptic, even for fully subsonic flow, but contains hyperbolic partitions. TME for the Navier–Stokes simulations can be achieved if the different factors contributing to the system could be separated and treated optimally, e.g., by multigrid for elliptic factors and by downstream marching for hyperbolic factors. The major difficulty in efficiently solving the Navier–Stokes equations is encountered with the inviscid (Euler) subset; thus we restrict ourselves to the Euler equations in this paper.

Through the last thirty years, multigrid has been applied to accelerate convergence in Euler and Reynolds-averaged Navier Stokes codes. These applications of multigrid generally are based on the unsteady equations using some temporal integrator as the smoother, combined with a full approximation scheme (FAS). In 1985, Mulder (6), used upwind-differencing and implicit time integration as the smoother. However, these approaches have resulted in poor multigrid efficiency. When applied to high Reynolds number flows over complex geometries, convergence rates are often worse than 0.99.

In 1985, according to Brandt (2) one of the major obstacles to obtain better multigrid performance for advection-dominated flows is that the coarse grid provides only a fraction of the needed correction for smooth error components. This obstacle can be removed by designing a solver that effectively distinguishes between the elliptic, parabolic, and hyperbolic (advection) factors of the system and treats each one appropriately.

Brandt presented an approach called “distributive relaxation” by which one can construct smoothers that effectively distinguish between the different factors of the operator. Using this approach, in 1993, Brandt and Yavneh (3) have demonstrated textbook multigrid for the incompressible Navier-

Stokes equations. Their results are for a simple geometry and a Cartesian grid, using a staggered-grid discretization of the equations.

In 1999, Brandt et al (12) presented a multigrid method that attains textbook efficiencies for one of the most basic simulations encountered in fluid mechanics-the incompressible viscous flow past a finite flat plate at high Reynolds number. The flow, although relatively simple, contains several basic elements of the barriers to be overcome in extending textbook efficiencies to the compressible equations, namely entering flows, far wake flows, and boundary layers. A central element of the multigrid method presented is the decomposition through distributed relaxation (2) of the system of equations into separate, usually scalar, factors that can be treated optimally, i.e., through marching for the hyperbolic factors and through multigrid for the elliptic factors.

In a closely related approach, Ta'asan (11) presents a fast multigrid solver for the compressible Euler equations. This method is based on a set of "canonical variables" which express the steady Euler equations in terms of an elliptic and a hyperbolic partition. Ta'asan uses this partition to guide the discretization of the equations. A staggered grid is used, with different variables residing at cell, vertex, and edge centers. In this reference it is shown that ideal multigrid efficiency can be achieved for the compressible Euler equations for two-dimensional subsonic flow using body fitted grids. One possible limitation of the use of the canonical variables is that the partition of the inviscid equations is not directly applicable to the viscous equations.

In 1997, Thomas Roberts, David Sidilkover, and Swanson (7) produced an alternative to distributive relaxation and to Ta'asan's canonical variable decomposition. It is a generalization of the approach of Sidilkover et al (9-10). This approach can be classified as a method of the weighted Gauss-Seidel type. A conventional vertex-based finite volume or finite difference discretization of the primitive variables is used, avoiding the need for staggered grids. This simplifies the restriction and prolongation operations, because the same operator can be used for all variables. A projection operator is applied to the system of equations resulting in a Poisson equation for the pressure. By applying the projection operator to the discrete equations rather than to the differential equations, the proper boundary condition on the pressure is satisfied directly. Because the elliptic and advection parts of the system are decoupled, ideal multigrid efficiency can be achieved. Compared to the distributive relaxation and the canonical variables' approaches, this method is extremely simple.

The rest of this paper is organized such that Sec. 1 presents the application of projection of the differential operator of the Euler equation yielding a Poisson's equation for pressure in place of the continuity equation. The required additional boundary condition for pressure is derived from given boundary condition of the velocity components. The discretization of the resulting system of differential equations is presented in Sec. 2. The details of the multigrid routines; relaxation, restriction, and prolongation as well as a V-cycle are presented in Sec. 3. In Sec. 4, the robustness and efficiency of the proposed method are demonstrated by presenting convergence analysis of the results obtained from solving the Euler equation on rectangular domain where the exact solution is known. Finally conclusions are given in Sec. 5.

NOMENCLATURE

(All Variables are in SI Units)

Notation	Explanation
FU, FV,FP	Right hand sides of the discretized Euler Equations (14)
h	Mesh size of a fine grid
ℓ	Number of grid levels
L	Differential operator
P	Pressure
Q	Advection operator
R	Projection operator
u	Velocity component in the x-direction
v	Velocity component in the y-direction
V	Multigrid V-cycle
Subscripts	
x	x derivative
y	y derivative
Superscripts	
m	current iteration
Abbreviations	
FAS	Full Approximation Scheme
FMG	Full Multigrid
MG	Multigrid
TME	Textbook Multigrid Efficiency

1- FORMULATION OF THE PROBLEM

The incompressible Euler equations in primitive variables in 2D are the two momentum equations in x, y directions and the continuity equation:

$$
\begin{aligned}
& u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial P}{\partial x} = 0 \\
& u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial P}{\partial y} = 0 \qquad (1) \\
& \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\end{aligned}
$$

where u and v are the components of the velocity in the x and y directions, respectively, and P is the pressure and the density is taken to be one.
Defining the advection operator as

$$Q \equiv u\partial_x + v\partial_y \qquad (2)$$

where ∂_x, ∂_y are the partial differentiation operators.

The Euler equations can be written as

$$Lq = \begin{pmatrix} Q & 0 & \partial_x \\ 0 & Q & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (3)$$

1.1 Pressure Poisson's Formulation
Introducing the operator Q^*, defined by

$$Q^*(f) = -\partial_x(uf) - \partial_y(vf) \quad , \qquad (4)$$

a projection operator R is defined as:

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_x & \partial_y & Q^* \end{pmatrix} \qquad (5)$$

Applying the projection operator to the Euler equations yields

$$\tilde{L}q \equiv RLq = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_x & \partial_y & Q^* \end{pmatrix}\begin{pmatrix} Q & 0 & \partial_x \\ 0 & Q & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, (6)$$

Or,

$$\begin{pmatrix} Q & 0 & \partial_x \\ 0 & Q & \partial_y \\ 0 & 0 & \Delta \end{pmatrix}\begin{pmatrix} u \\ v \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \partial_x Q(u) + Q^*(u_x) + \partial_y Q(v) + Q^*(v_y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (7)$$

Simplifying the third row of the right matrix, Euler equations becomes:

$$\begin{pmatrix} Q & 0 & \partial_x \\ 0 & Q & \partial_y \\ 0 & 0 & \Delta \end{pmatrix}\begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2u_x v_y - 2u_y v_x \end{pmatrix} \qquad (8)$$

where $\Delta = \partial_{xx} + \partial_{yy}$ is the Laplacian operator. The matrix operator on the left-hand side of Eq. (8) is upper triangular. The pressure satisfies a Poisson equation for which a conventional relaxation method, such as Gauss-Seidel, can be applied. The momentum equations (first and second rows of matrix Eq.(8)) can be looked at as a standard advection equations with known pressure gradients and hence upwind differencing allows downstream relaxation to be used (5).
It is important to discuss the boundary conditions for the pressure. It is noticed that the order (2) of Eq.8 (second partial derivatives for P) is higher than the original system Eq.1, where only first derivatives were required. Therefore, to ensure the well posedness of the problem, an extra boundary condition is essential. This additional boundary condition needs to be derived from the boundary conditions specified for the original problem and, possibly, differential equations of the original system (Eq.1).

1.2 Treating the Boundary Conditions on a Rectangular Grid

In case of a rectangular domain the boundary conditions can be treated easily. At a horizontal boundary y=b, if the velocity components u(x, b) and v(x, b) are known, then their x-derivatives u_x and v_x can be computed. So, p_y can be obtained using the second momentum equation and the continuity equation as follows:

(5)

$$P_y = -uv_x - vv_y = -uv_x + vu_x \qquad (9)$$

Similarly, at a vertical boundary x=a, if the velocity components u(a,y) and v(a,y) are known, then their y-derivatives u_y and v_y can be obtained and hence a boundary condition for pressure is derived using the first momentum equation and the continuity equation as follows:

$$P_x = -uu_x - vu_y = uv_y - vu_y \qquad (10)$$

However the required additional conditions for pressure associated with other arbitrary boundary conditions on even curved boundaries can be derived (6).

2- DISCRETIZATION OF EULER EQUATIONS ON A RECTANGLE

To build the discrete scheme, the momentum equations are discretized using standard first-order upwind-difference approximation to the advection operator and the pressure gradient. However second-order central-difference approximation is used to discretize the Laplacian operator. If a uniform mesh with mesh size h is used to discretize rectangular domain ($0 \le x \le Ih, 0 \le y \le Jh$), the discrete system for the Euler equations (8) at an interior node with coordinates (ih, jh), 0<i<I, 0<j<J is given by:

(8)

$$\left.\begin{aligned}
&u_{i,j}\frac{u_{i,j}-u_{i-1,j}}{h}+v_{i,j}\frac{u_{i,j}-u_{i,j\text{-}1}}{h}+\frac{p_{i,j}-p_{i-1,j}}{h}=0\\
&u_{i,j}\frac{v_{i,j}-v_{i-1,j}}{h}+v_{i,j}\frac{v_{i,j}-v_{i,j\text{-}1}}{h}+\frac{p_{i,j}-p_{i,j\text{-}1}}{h}=0\\
&\frac{p_{i+1,j}+p_{i-1,j}+p_{i,j+1}+p_{i,j-1}-4p_{i,j}}{h^2}=\\
&2\frac{u_{i,j}-u_{i-1,j}}{h}\cdot\frac{v_{i,j}-v_{i,j-1}}{h}-2\frac{u_{i,j}-u_{i,j-1}}{h}\cdot\frac{v_{i,j}-v_{i-1,j}}{h}
\end{aligned}\right\}(11)$$

where $s_{i,j}$ is the s-variable at node (i,j). Thus Eq.11 is a system of three nonlinear algebraic equations in u, v and p at node (i,j) and some of its neighboring nodes. This system has to be modified at boundaries appropriately. For example, at a lower horizontal boundary (j=0), if velocity components are specified, the value of P_y at that boundary is computed according to Eq.9. The discrete form of this boundary condition is given by:

$$P_y\Big|_{i,j}=\frac{P_{i,j+1}-P_{i,j-1}}{2h}+O\left(h^2\right) \qquad (12)$$

Equation 12 is used to substitute for the pressure at the false node (*i,j-1*) and the third equation in Eq.11 reduces to:

$$\begin{aligned}
&\frac{P_{i+1,j}+P_{i-1,j}+2P_{i,j+1}-2h\cdot P_y\Big|_{i,j}-4P_{i,j}}{h^2}=\\
&2\frac{u_{i,j}-u_{i-1,j}}{h}\cdot\frac{v_{i,j}-v_{i,j-1}}{h}-2\frac{u_{i,j}-u_{i,j-1}}{h}\cdot\frac{v_{i,j}-v_{i-1,j}}{h}
\end{aligned}\qquad(13)$$

3- THE MULTIGRID TECHNIQUE FOR SOLVING EULER EQUATIONS

The FAS multigrid technique will now be implemented for solving the discretized Euler equations (Eq.10). The main steps of the FAS-multigrid V-cycle will be discussed in details.

The basic step in developing an efficient multigrid algorithm is to design an efficient relaxation procedure. For nonlinear problems, the relaxation updates to a current solution approximation are usually computed through Newton iterations. The full Newton linearization method is a complicated operator, and its solution is expensive. That is because the objective of the relaxation in MG solvers is to smooth the error and not to eliminate it. Instead, the principal linearization is adopted here (2). The discretized highest derivative terms are considered principal and only their updates are considered through relaxation.

If superscripts are used to distinguish between the variables that will be updated in a current relaxation step 'm', and those that will be substituted their values from the previous relaxation step 'm-1', then the discrete system is linearized as:

$$\left.\begin{aligned}
&u^{m\text{-}1}_{i,j}\frac{u^{m}_{i,j}-u^{m-1}_{i-1,j}}{h}+v^{m\text{-}1}_{i,j}\frac{u^{m}_{i,j}-u^{m-1}_{i,j\text{-}1}}{h}+\frac{p^{m\text{-}1}_{i,j}-p^{m-1}_{i-1,j}}{h}=0\\
&u^{m\text{-}1}_{i,j}\frac{v^{m}_{i,j}-v^{m-1}_{i-1,j}}{h}+v^{m\text{-}1}_{i,j}\frac{v^{m}_{i,j}-v^{m-1}_{i,j\text{-}1}}{h}+\frac{p^{m\text{-}1}_{i,j}-p^{m-1}_{i,j\text{-}1}}{h}=0\\
&\frac{p^{m\text{-}1}_{i+1,j}+p^{m-1}_{i-1,j}+p^{m-1}_{i,j+1}+p^{m-1}_{i,j-1}-4p^{m}_{i,j}}{h^2}=\\
&2\left(\frac{u^{m-1}_{i,j}-u^{m-1}_{i-1,j}}{h}\cdot\frac{v^{m-1}_{i,j}-v^{m-1}_{i,j-1}}{h}-\frac{u^{m-1}_{i,j}-u^{m-1}_{i,j-1}}{h}\cdot\frac{v^{m-1}_{i,j}-v^{m-1}_{i-1,j}}{h}\right)
\end{aligned}\right\}(14)$$

It must be noted that, according to the used MG-cycle, one or two relaxation steps are required (m=1 or 2) and that initial values of the variables are supplied either by the user or from previous calculations. For the purpose of relaxation, at the point (i,j), the first and second momentum equations in system (14) are used for updating the u- and v-velocity components, and the Poisson equation is used for updating the pressure. Also to design a relaxation scheme for all grids, general right hand sides are introduced and Eq.14 is rearranged as:

$$\left.\begin{aligned}
&u^{m}_{i,j}=\frac{\left(FU_{i,j}\cdot h+u^{m\text{-}1}_{i,j}\cdot u^{m-1}_{i-1,j}+v^{m\text{-}1}_{i,j}\cdot u^{m-1}_{i,j\text{-}1}-p^{m\text{-}1}_{i,j}+p^{m-1}_{i-1,j}\right)}{\left(u^{m\text{-}1}_{i,j}+v^{m\text{-}1}_{i,j}\right)}\\
&v^{m}_{i,j}=\frac{\left(FV_{i,j}\cdot h+u^{m\text{-}1}_{i,j}\cdot v^{m-1}_{i-1,j}+v^{m\text{-}1}_{i,j}\cdot v^{m-1}_{i,j\text{-}1}-p^{m\text{-}1}_{i,j}+p^{m-1}_{i,j\text{-}1}\right)}{\left(u^{m\text{-}1}_{i,j}+v^{m\text{-}1}_{i,j}\right)}\\
&p^{m}_{i,j}=\frac{\left(p^{m\text{-}1}_{i+1,j}+p^{m-1}_{i-1,j}+p^{m-1}_{i,j+1}+p^{m-1}_{i,j-1}-FP_{i,j}\cdot h^2\right)}{4}
\end{aligned}\right\}(15)$$

where, for the computational grid,

$$\begin{aligned}
&FU_{i,j}=0,\ FV_{i,j}=0,\\
&FP_{i,j}=2\left(\frac{u^{m\text{-}1}_{i,j}-u^{m\text{-}1}_{i-1,j}}{h}\cdot\frac{v^{m\text{-}1}_{i,j}-v^{m\text{-}1}_{i,j-1}}{h}-\frac{u^{m\text{-}1}_{i,j}-u^{m\text{-}1}_{i,j-1}}{h}\cdot\frac{v^{m\text{-}1}_{i,j}-v^{m\text{-}1}_{i-1,j}}{h}\right)
\end{aligned}\qquad(16)$$

On the other hand, $FU_{i,j}$, $FV_{i,j}$, $FP_{i,j}$ are computed on coarser grids through the FAS restriction subroutine.

The Gauss-Seidel relaxation is applied in lexicographic order in the downstream direction. This means that the nodes variables are updated sequentially according to their order in

the flow direction and beginning from nodes with known velocity. So this order is dependent on the boundary conditions. Downstream marching is a very efficient solver that yields an accurate solution to a nonlinear hyperbolic equation(5). For the convection terms in the momentum equations a single downstream sweep provides the exact solution to the linearized problem (first and second equations in Eq.14) provided that the pressure gradient is exact.
It is important to notice that while Eq.14 is used for computing relaxation updates, the original formulation Eq.13, with appropriate right hand sides, is used for computing residuals. For the transfer of residuals from fine to coarse grids, two methods of restriction will be implemented in this work; the trivial injection and the full weighting operators. For the prolongation, transfer of corrections from coarse to fine grids, bilinear interpolation is adopted (1).

4- NUMERICAL EXAMPLE AND RESULTS

The problem of inviscid flow over a rectangular domain is considered here. This problem is formulated by the Euler's equations (Eq.1) within the rectangle and proper boundary conditions.

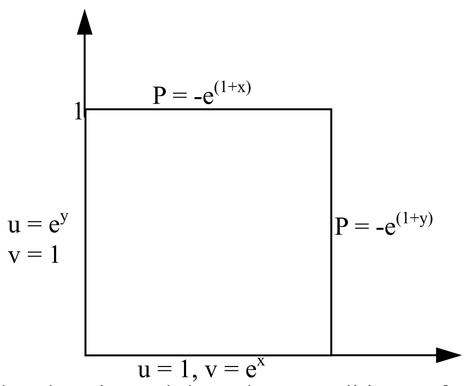

Fig.1. The domain and boundary conditions of the model problem

Figure (1) shows the domain, dimensions, and boundary conditions of the model example. It is easy to prove that this problem has the exact solution:

$$u = e^{y} \quad , \quad v = e^{x} \quad , \quad p = -e^{x+y} \qquad (17)$$

A nested sequence of uniform grids is constructed with mesh sizes $h = \frac{1}{2^{\ell}}$, $\ell = 1, 2, \cdots, L$. For the coarsest grid G_1, $h = \frac{1}{2}$, the grid has 9 nodes while on the finest grid, $h = \frac{1}{2^L}$, there are $(2^L + 1)^2$ nodes. The ordering of the grid vertices was from the lower-left to the upper–right of the domain resulting in downstream relaxation. A FAS_V(2,1) multigrid cycle is used; that is, two relaxation sweeps are performed on each grid before restriction to a coarser grid, and one relaxation sweep is performed after prolongation to a fine grid. If one work unit is defined as the computational work required to perform a Gauss-Seidel relaxation on the finest grid, a single V(2,1) cycle is approximately 4 work units.

Before discussing the convergence behavior of the proposed approach it is important to remember the following definitions. The algebraic error is defined as the difference between the exact and approximate *solutions of the discrete problem*. Practically, the residue induced when approximate solution is substituted in the discrete system, is computed rather than the algebraic error since the exact solution of the algebraic system is generally unknown. On the other hand, a fast residual convergence is considered as an important monitoring tool for the algebraic error.

4.1 Comparison Between Multigrid and Single Grid

The Gauss-Seidel relaxation described in the previous section is used to solve the inviscid flow problem on a uniform grid G_6 with mesh size h=1/64. After each relaxation, the L_2-residual norm is computed for u,v, and P. The same problem is solved by multigrid V(2,1)-cycle using the same Gauss-Seidel relaxation scheme and the direct injection operator to restrict both the variables and residuals. The L_2- residual norm is computed for u,v, and Pat the end of each cycle. The results are plotted on Fig. 2 for the v-component (similar convergence behavior is obtained for u) and on Fig 3 for pressure. Although the proposed Gauss-Seidel relaxation on the single grid converges, the convergence rates slowdown quickly after few relaxations. This performance is commonly observed for all single grid iterative solvers. The convergence rate of multigrid is uniform and is relatively very efficient. However, it is still possible to increase these MG convergence rates.

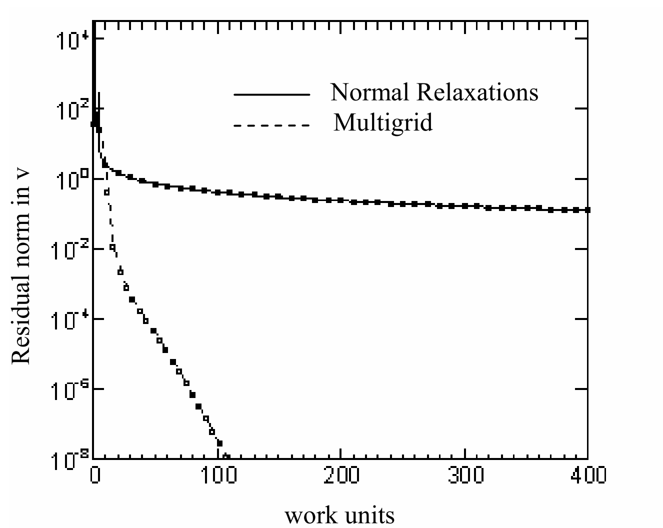

Fig.2. Convergence rates of v for multigrid and single grid (Gauss-Seidel)

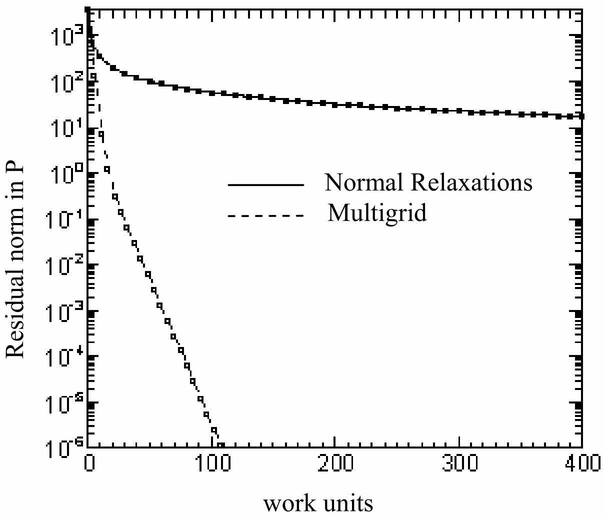

Fig.3. Convergence rates of P for multigrid and single grid (Gauss-Seidel)

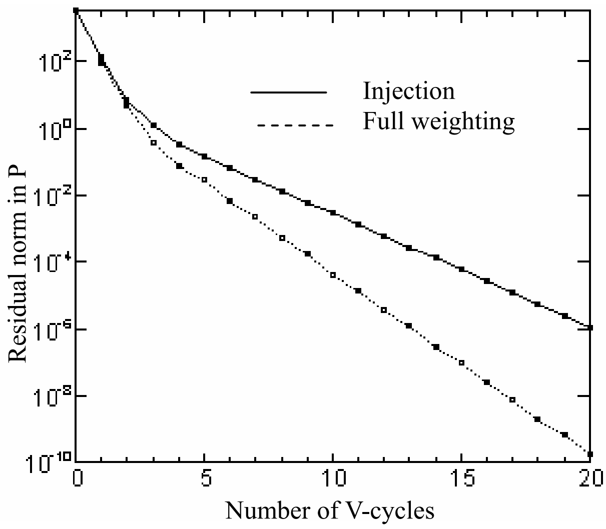

Fig.5. MG convergence rates for P using: injection and full weighting operators.

4.2 Using The Full Weighting Restriction Operator

The full weighting restriction operator is implemented in the algorithm instead of the injection one. In Figs 4 and 5, the convergence rates of the residual norms of the two proposed algorithms are presented for the v-velocity component (u-velocity component is similar), and the pressure. The problem is solved on G_6 computational grid, i.e. h=1/64.

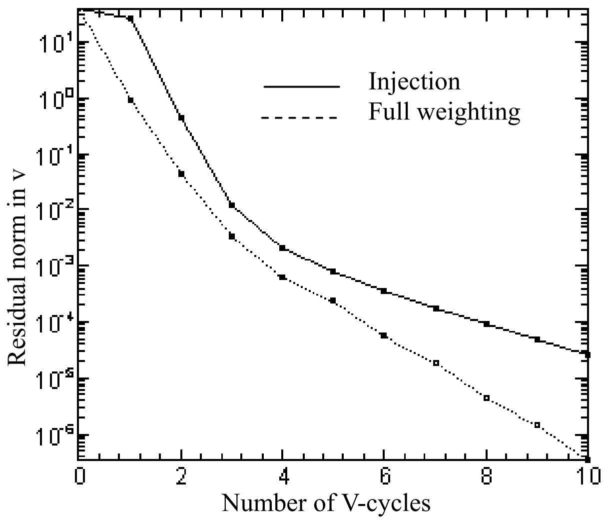

Fig.4. MG convergence rates for v using: injection and full weighting operators.

4.3 The Dependency of the Convergence Rate on the Mesh Size

Previous results concerning convergence rates were performed on a computational grid G_6. h=1/256. To discuss the dependency of the convergence rate on the mesh size of the computational grid, the convergence rates are computed when the problem is solved on different computational grids with mesh sizes h=1/32, h=1/64, h=1/128, h=1/256 As shown in Figs. 6 and 7, the MG convergence rates are independent of the mesh size. This feature of the MG convergence rates ensures the robustness of the proposed algorithm.

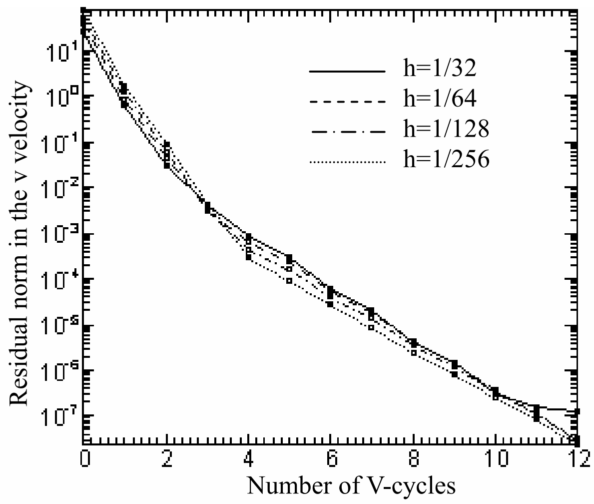

Fig.6. Convergence rates of v for different computational grids

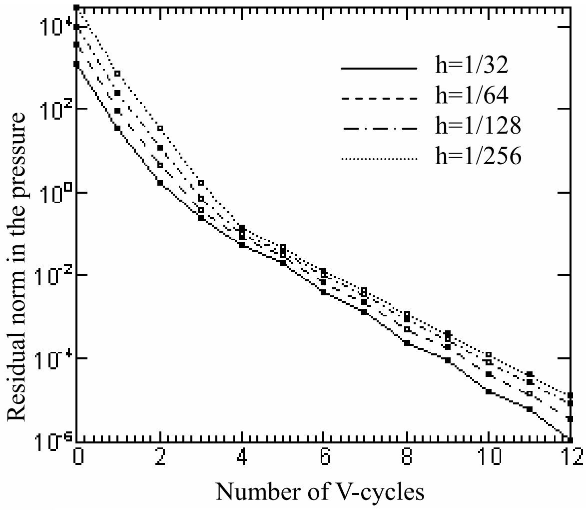

Fig.7. Convergence rates of P for different computational grids

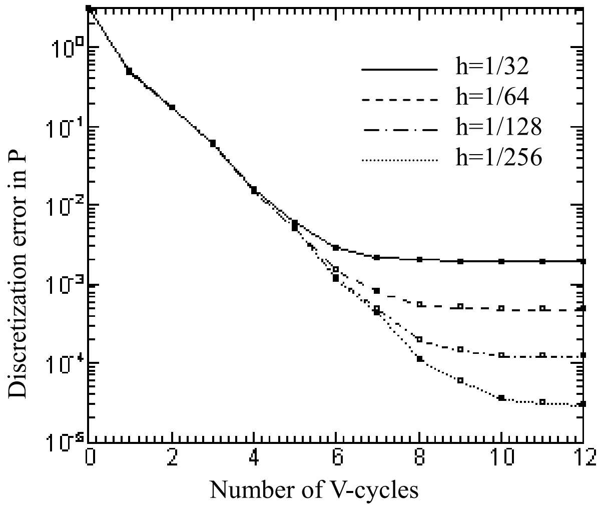

Fig.9. Discretization error in pressure

4.4 The Discretization Error

In reality, the objective of MG algorithms is fast convergence to the *solution of the differential equations*, not necessarily fast asymptotic residual convergence. The natural solution tolerance is the discretization error defined as the difference between the exact solutions of discrete and differential problems. To discuss the behavior of the proposed algorithm and since the exact solution is known (Eq.17), the L_2-norm of the difference between computed solution after each cycle and the exact solution of the differential system is calculated and plotted in Fig.8 for the v-velocity component and
in Fig.9 for the pressure. These results are presented for different computational grids with mesh sizes h=1/32, h=1/64, h=1/128, h=1/256.

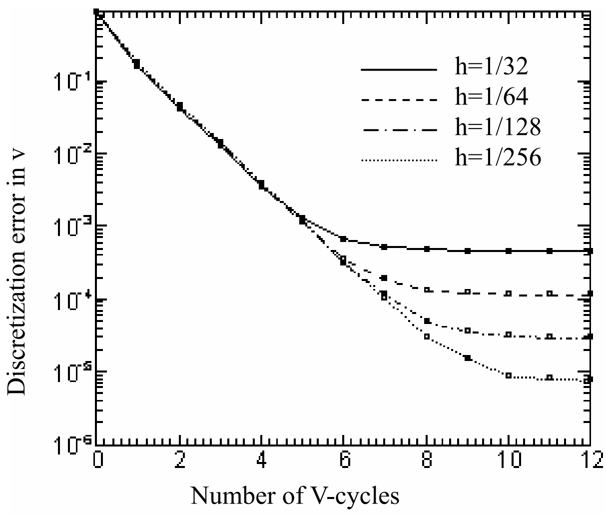

Fig.8. Discretization error in v-velocity component

It is observed that, for a given computational grid, this computed norm decreases uniformly through the first few cycles (4-10 cycles) then remains unchanged at the value of the discretization error at this grid. This means that excess computation does not increase the objective accuracy. That is because the exact solution of the discrete system has been reached (within error less than the dicretization error) after these few cycles. The L_2-norms of the discretization error in u,v, and P after complete convergence are shown in table 1 for various grid sizes.

Table (1) Discretization error of the unknowns: u, v, P for various computational grids.

	u	v	P
h=1/32	4.63E-04	4.62E-04	1.93E-03
h=1/64	1.20E-04	1.20E E-04	4.87E-04
h=1/128	3.07E-05	3.07E-05	1.23E-04
h=1/256	7.82E-06	7.82E-06	3.09E-05

5. CONCLUSIONS

A multigrid method for solving the incompressible inviscid Euler equation has been developed by exploiting factorizability of the governing differential operator. Treating each of the factors appropriately, optimal convergence rates have been attained. First order discretization scheme is used for the momentum equations while a second order finite difference scheme for Poisson equation. No artificial viscosity is introduced in the discretization. Elements of the Full Approximation Scheme multigrid algorithm, including relaxation, residuals, restriction, prolongation, cycle, and boundary treating have been presented in some detail. The algorithm is tested by solving a model problem with known exact solution. The main conclusions are:

1. After few (4-10) multigrid V-cycles, the algebraic error is reduced substantially below the discretization error.
2. Full-weighting restriction operator is more efficient than the injection one.
3. The convergence rates are independent of the mesh size.
4. Lexicographic Gauss-Seidel using downstream ordering shows excellent smoothing rates. But it is complicated to maintain downstream ordering in case the flow directions change with location.

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