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ITERATIVE SOLVERS AND INFLOW BOUNDARY CONDITIONS FOR PLANE SUDDEN EXPANSION FLOWS

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ABSTRACT

Incompressible laminar flow in a symmetric plane sudden expansion is studied numerically. The flow is known to exhibit a stable symmetric solution up to a critical Reynolds number above which symmetry-breaking bifurcation occurs. The aim of the present study is to investigate the effect of using different iterative solvers on the calculation of the bifurcation point. For this purpose, the governing equations for steady two-dimensional incompressible flow are written in terms of a stream function-vorticity formulation. A second order finite volume discretization is applied. Explicit and implicit solvers are used to solve the resulting system of algebraic equations. It is shown that the explicit solver recovers the stable asymmetric solution, while the implicit solver can recover both the unstable symmetric solution or the stable asymmetric solution, depending on whether the initial guess is symmetric or not. It is also found that the type of inflow velocity profile, whether uniform or parabolic, has a significant effect on the onset of bifurcation as uniform inflows tend to stabilize the symmetric solution by delaying the onset of bifurcation to a higher Reynolds number as compared to parabolic inflows.

INTRODUCTION

Bifurcation phenomena are frequently encountered in fluid mechanics. Some examples include transonic inviscid flows over airfoils [1] and cylinders [2]-[3], incompressible and compressible viscous flows over airfoils [4]-[5] and incompressible viscous flows through plane sudden expansions. The latter is the subject of the present paper.

Incompressible laminar flow in a symmetric plane sudden expansion is a classical fluid flow problem which admits multiple solutions. This fact has been demonstrated both numerically and experimentally by several authors. For Reynolds numbers less than a certain critical value, the flow in the sudden expansion is symmetric with two equal sized eddies whose length increases linearly with Reynolds number. For Reynolds numbers higher than the critical value, the flow undergoes a symmetry-breaking pitchfork bifurcation rendering the symmetric solution unstable. The stable solution becomes asymmetric with two eddies of different sizes. Two asymmetric solutions are possible, with one being the mirror image of the other.

The study of bifurcation in fluid mechanics has received considerable interest as it enables a better understanding of the problems of stability and laminar-to-turbulent flow transition. Experimental investigations of the flow in a sudden expansion include the work of Durst, Melling & Whitelaw [6], Cherdron, Durst & Whitelaw [7] and Ouwa, Watanabe & Asawo [8]. These experimental results show that the flow in the sudden expansion remains symmetric at low Reynolds numbers, but becomes asymmetric at higher values. Numerical computations of the symmetry-breaking bifurcation point by Fearn et al [9] and Durst et al [10], together with the linear stability analysis of symmetric flow in a plane sudden expansion by Shapira et al [11] indicate that this observed experimental behavior occurs at a bifurcation of the Navier-Stokes equations.

Other contributions regarding the symmetry-breaking pitchfork bifurcation include the investigations of Rusak and Hawa. In [12], Rusak and Hawa carried out a weakly nonlinear analysis of the bifurcation, and in [13] they studied the effect of a slight asymmetry of the channel geometry on the flow behavior. The structural instability of the bifurcation was also studied with a weakly nonlinear stability analysis by Mizushima and Shiotani [14].

The dependence of the critical Reynolds number of the symmetry-breaking bifurcation on the expansion ratio was studied numerically by Allerborn et al.[15], Battaglia et al.[16] and Drikakis [17]. It was found that reducing the expansion ratio tends to improve the stability of the symmetric solution. Bifurcation of plane sudden expansion flows in the case of large expansion ratios was studied by Revuelta [18]. Moreover, turbulent flow asymmetries through plane symmetric expansions were reported by Abbott and Kline [19], while non-newtonian flow asymmetries through plane sudden expansions were calculated by Neofytou and Drikakis [20] and also by Manica and De Bortoli [21].

In all the previous studies, the inflow velocity profile was parabolic, corresponding to a fully developed flow in the upstream channel. However, if the sudden expansion is located within the entrance length of the upstream channel, the inflow would certainly not be parabolic. Hence, plane sudden expansion flows could be subjected to different types of inflow profiles. No studies, however, investigated the effect of the type of inflow profile on the onset of bifurcation. In this paper, both uniform and parabolic inflows are considered, and the effect of the type of inflow profile on the critical Reynolds number and on the flow field is clarified. Also, the effect of using different types of solvers, explicit and implicit, on the calculation of the critical Reynolds number is investigated. Finally, all three possible solutions, the two stable asymmetric solutions and the unstable symmetric solution, are recovered in order to demonstrate the profound effect of the solver choice on the recovered solution.

NOMENCLATURE

ψ	Stream function
ω	Vorticity
u	x-component of velocity
v	y-component of velocity
u_{av}	Average velocity in the upstream channel
d	Upstream channel height
D	Downstream channel height
Q	Volume flow rate
ν	Kinematic viscosity
ρ	Fluid density
τ_w	Wall shear stress
c_f	Skin friction coefficient
Re	Reynolds number= $u_{av}\, d/\nu$
Re_{cr}	Critical Reynolds number at the bifurcation point
L_r	Recirculation bubble length

1. Governing Equations and Numerical Methods

In the present study, the Navier–Stokes equations for two-dimensional steady incompressible flow are written in terms of a stream function-vorticity formulation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega \qquad (1)$$

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \qquad (2)$$

The expansion ratio is defined as the ratio of downstream channel height to upstream channel height (D/d). Due to the moderate range of Reynolds numbers considered in this study (Re<1000), it is suffice to use a second order accurate central-based finite volume scheme for all terms, including the inviscid flux terms of the vorticity transport equation. In this context, it should be mentioned that Sobey and Mullin [22] reported that central-based schemes predict a bifurcation structure closer to experimental observations than upwind schemes.

The formulation is solved in a segregated manner. For the Poisson's equation of the stream function, two types of iterative solvers are considered. The first is a point successive over-relaxation (PSOR) scheme. The second is a line successive over-relaxation (LSOR) scheme in which the grid lines normal to the main flow direction are solved implicitly using a tri-diagonal solver. For both iterative solvers, the flow domain is swept repeatedly along the main flow direction till convergence. For the vorticity transport equation, a time marching procedure is used, with an artificial time dependent term being added to the equation to improve numerical stability. Again, two types of iterative solvers are considered for the vorticity transport equation. The first is the classical explicit fourth order Runge-Kutta scheme while the second is a line implicit scheme in which the grid lines normal to the main flow direction are solved implicitly using a tri-diagonal solver.

Two different combinations of these iterative solvers are considered. The first combination, termed solver (1), is an explicit solver which uses PSOR for the Poisson's equation of the stream function and the fourth order Runge-Kutta scheme for the vorticity transport equation. The other combination, termed solver (2), is an implicit solver which uses LSOR for the stream function equation and a line implicit solver for the vorticity transport equation.

For an accurate prediction of the separation and re-attachment points, the skin friction, c_f, is evaluated

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho u^2_{av}} \quad (3)$$

A one-sided second-order accurate difference formula is used in the evaluation of the skin friction on the upper and lower walls of the downstream channel.

2. Inflow Boundary Condition

The incompressible laminar flow in a sudden expansion of ratio 4:1 is investigated. The explicit solver, solver (1), is used with a symmetric initial guess in the flow domain. In most of the previous studies, only the downstream channel was included in the flow domain and a parabolic inflow was enforced at the expansion plane. Oliviera and Pinho [23] showed that for low Reynolds number, the velocity distribution deviates slightly from the parabolic profile at the expansion plane for the case of an axisymmetric sudden expansion.

Therefore, in the present study and to exclude any assumptions regarding the shape of the velocity profile at the expansion plane, a two-block rectangular grid is used. The first block includes the upstream channel while the second block includes the downstream channel, with mass and momentum being conserved across the interface between the two blocks.

For the parabolic inflow, a parabolic velocity distribution is enforced at the inlet of the upstream channel and the no-slip condition is applied to the walls of the upstream channel (viscous wall boundary condition). For the uniform inflow, a uniform velocity distribution is enforced at the inlet of the upstream channel and a no-penetration condition is applied to the walls of the upstream channel (inviscid wall boundary condition). As for the walls of the downstream channel, the no-slip condition (viscous wall boundary condition) is applied for both parabolic and uniform inflows. The length of the downstream channel is selected such that a fully developed Poiseuille flow is recovered at the downstream end. Extrapolation is used as a boundary condition for vorticity at the downstream end.

To confirm that the numerical solution is grid-independent, the flow in a sudden expansion with a parabolic inflow velocity profile, at Re=30, is solved using two different grids. The characteristics of the two blocks of the first grid are (first block: 100 X 21, second block: 300 X 81) while those for the second grid are (first block: 200 X 41, second block: 600 X 161). The velocity profiles at various locations, resulting from both grids, are compared in fig (1), confirming that the solution is grid-independent. In fig (1), (x) represents the distance measured from the expansion plane. Also, after some re-developing length downstream of the expansion plane, fig (1) shows that the numerical solutions on both grids recover the fully developed parabolic velocity distribution.

After confirming that the solution is grid-independent, the effect of increasing Reynolds number is investigated. For low Reynolds numbers, the solution is symmetric with two eddies of equal sizes. The size of eddies increases linearly with increasing Reynolds number as can be seen from fig (2). For the parabolic inflow, the re-attachment length calculated in the present work agrees well with the correlation introduced by Scott et al. [24],

$$\frac{L_r}{d} = 0.178\,\mathrm{Re} \quad (4)$$

which is based on the numerical results of their finite element analysis of the incompressible laminar flow through a sudden expansion of ratio 4:1.

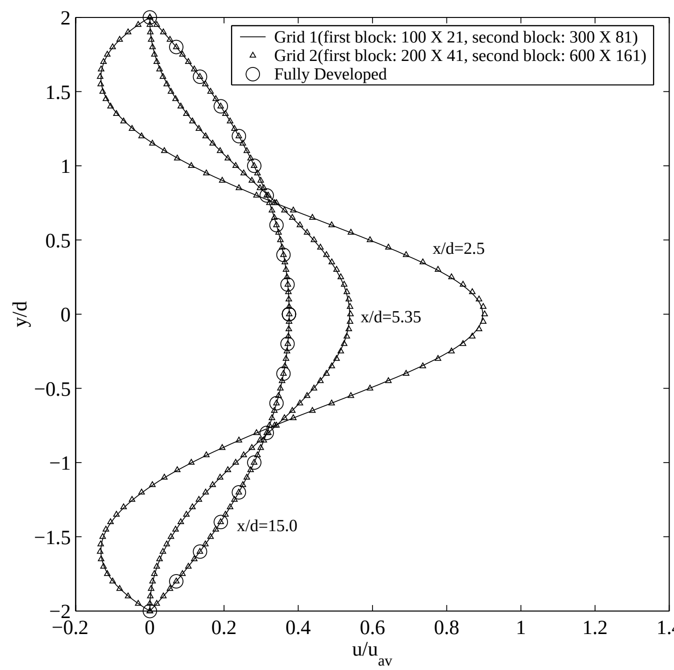

Fig.1- Grid independence test (Re=30)

As evident from fig (2), uniform inflow results in a smaller re-circulation region when compared with that of parabolic inflow for the same Reynolds number.

It should be noted, however, that the correlation of Scott et al. is only valid for high Reynolds number (Re>1) while it becomes asymptotically invalid in the limit of creeping flows (Re=0). This is demonstrated in fig (3), where the streamlines for the creeping flow limit (Re=0) are plotted for the case of a parabolic inflow in a 4:1 plane sudden expansion. Finite sized eddies are clearly present in the flow field with a dimensionless length of (L_r/d=0.725), while Scott et al.'s correlation predicts zero re-circulation length for the (Re=0) case. The appearance of eddies in creeping flows near sharp corners was originally studied by Moffat [25].

Fig (3) provides another interesting feature for the (Re=0) case, as it shows that the separation point does not coincide with the sharp corner, but rather lies at a point along the vertical wall at some distance away from the sharp corner. A similar behavior was reported by Hawa and Rusak [26] for the case of (Re=5), where the separation point lied along the vertical wall at a distance of (0.1d) from the sharp corner.

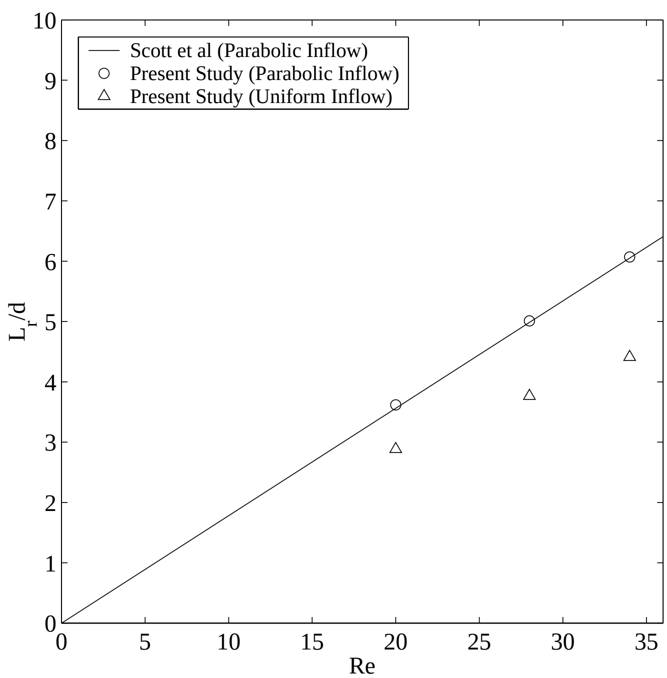

Fig.2- Recirculation Bubble Length versus Reynolds Number

As the Reynolds number is increased, a critical value is reached at which symmetry-breaking bifurcation occurs and the symmetric solution is no longer stable. In order to determine the value of the critical Reynolds number, Re_{cr}, symmetric and asymmetric flows on either side of the bifurcation point are computed yielding an ever-decreasing range of Reynolds numbers in which the bifurcation must occur. It should be noted, however, that as the simulated Reynolds number approaches the critical value, the simulations become computationally very expensive.

Bifurcation diagrams for uniform and parabolic inflows are shown in figs (4) and (5) respectively. In these figures, the distance between the reattachment points on the lower and upper walls (Δx) is plotted against the Reynolds number. The critical Reynolds number for the parabolic inflow profile is found to be 36, which is in good agreement with the bifurcation calculations of Battaglia et al. [16] which predicted a critical Reynolds number of (Re_{cr}=35.8).

However, for the uniform inflow profile, the flow is able to maintain a stable symmetric solution up to a critical Reynolds number of 64, as evident from fig (4). This clearly shows that the uniform inflow tends to delay the onset of bifurcation, resulting in an improvement in the stability of the symmetric solution. This result could be partially attributed to the uniform inflow having a lower momentum flux than the parabolic inflow for the same volume flow rate.

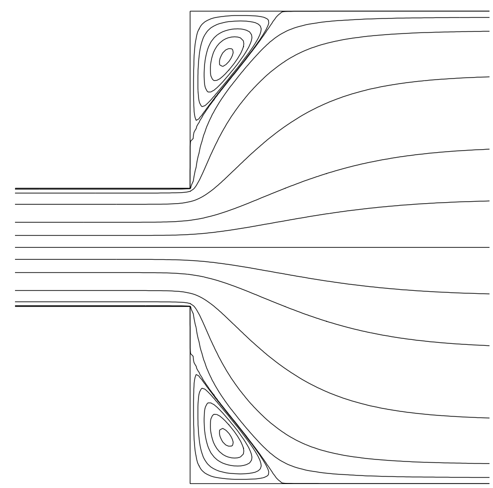

Fig.3- Streamlines for the zero Reynolds Number limit

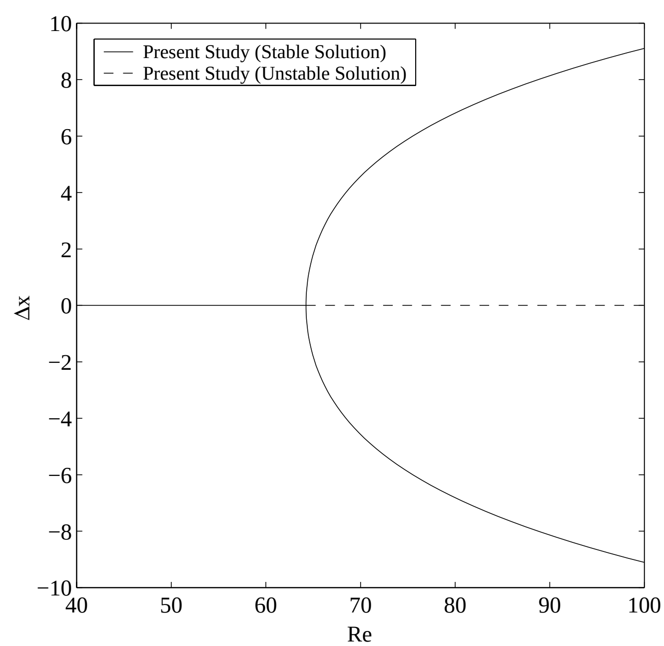

Fig.4- Bifurcation diagram for uniform inflow

whether the initial guess is symmetric or not. On the contrary, the solution obtained from the implicit solver is found to be highly dependent on the initial guess. When a symmetric initial guess is used, the implicit solver recovers the unstable symmetric solution. To recover one of the two stable asymmetric solutions using the implicit solver, an asymmetric initial guess has to be used.

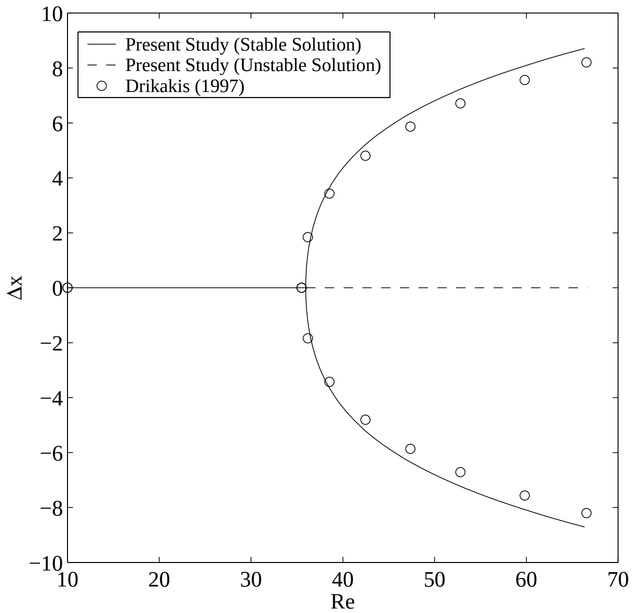

Fig.5- Bifurcation diagram for parabolic inflow

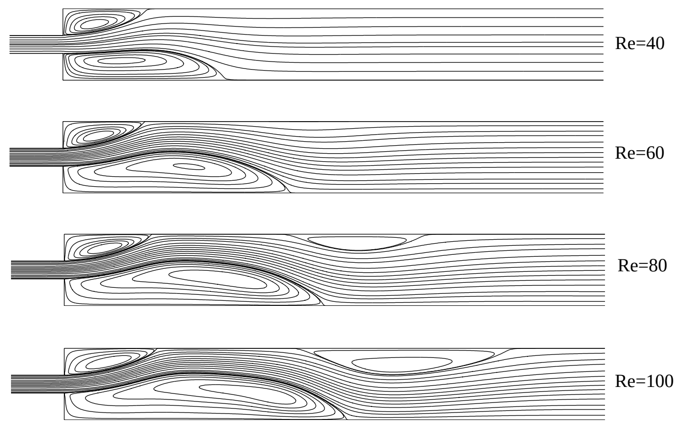

Fig.6- Streamlines for parabolic inflow at different Reynolds numbers

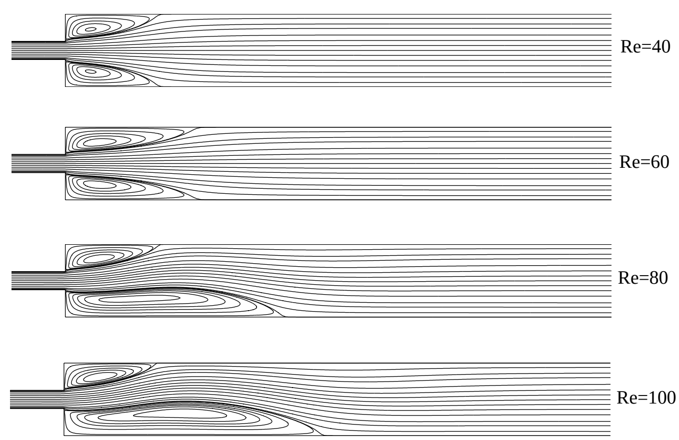

Fig.7- Streamlines for uniform inflow at different Reynolds numbers

Moreover, to demonstrate the profound effect the inflow has on the flow field itself, the streamlines for parabolic and uniform inflows at different Reynolds numbers are plotted in figs (6) and (7) respectively. For Re=40 and 60, the uniform inflow still maintains a stable symmetric solution, while the symmetric solution is no longer stable for the parabolic inflow, and the asymmetric solution prevails. For Re=80 and 100, a third re-circulating region appears for the parabolic inflow, while only two re-circulating regions are present for the uniform inflow.

Similar calculations are performed for other expansion ratios. The results are given in table (1), which also includes the results of Drikakis [17] for the sake of comparison. Table (1) confirms that uniform inflows result in higher critical Reynolds numbers for all expansion ratios considered and that the current results for parabolic inflows are in good agreement with the results of Drikakis [17]. Moreover, the effect of the type of inflow on the critical Reynolds number is shown to be much more significant for low expansion ratios, while it becomes of less importance in case of high expansion ratios as evident from Table (1).

3. Explicit and Implicit Solvers

In this section, solvers (1) and (2) are used to examine their effect on the numerical solution. The effect of using different initial guesses, symmetric and asymmetric, is also investigated.

When the explicit solver, solver (1), is used, one of the two stable asymmetric solutions is always recovered independent of

These interesting findings could be attributed to the explicit solver not retaining symmetry on the discrete level, with respect to the direction normal to the main flow, which inherently introduces the numerical instability required to shift the solution from the unstable symmetric branch to one of the two stable asymmetric branches. This, in turn, leads to the recovery of one of the two stable asymmetric solutions, even if started with a symmetric initial guess.

On the other hand, solver (2) is characterized by its implicitness, and hence symmetry on the discrete level, with respect to the direction normal to the main flow; hence no numerical instability is introduced through the solver. Therefore, when a symmetric initial guess is used in conjunction with solver (2), the unstable symmetric solution is successfully recovered. However, if an asymmetric initial guess is used with solver (2), one of the two stable asymmetric solutions is recovered, while the other asymmetric solution could also be recovered by starting with the mirror image of the asymmetric initial guess previously considered.

Table.1- Critical Reynolds numbers for different expansion ratios

Expansion Ratio	Re_{cr} (Parabolic Inflow)		Re_{cr} (Uniform Inflow)
	Drikakis (1997)	Present Study	Present Study
2:1	144	145	996
4:1	35	36	64
8:1	19	20	24

As a result of the above discussion, it is possible to recover all three possible solutions for flow in a sudden expansion depending on the type of solver selected. An example is given in figs (8) and (9) for a plane sudden expansion of ratio (4:1), where all three possible solutions for Re=100 are shown for parabolic and uniform inflows respectively through the use of the different iterative solvers and initial guesses. The unstable symmetric solutions in figs (8) and (9) are recovered using the implicit solver in conjunction with a symmetric initial guess. One of the two asymmetric solutions is obtained through the use of the explicit solver, while the other asymmetric solution is obtained using the implicit solver in conjunction with an appropriate asymmetric initial guess.

Moreover, skin friction distributions in the downstream channel of a plane sudden expansion of ratio (4:1) are given in fig (10).The skin friction distribution reveals that the flow regains symmetry after some re-developing length downstream of the expansion plane, at which the skin friction attains its fully developed value of

$$c_f = \frac{12}{Re}\left(\frac{d}{D}\right)^2 \tag{5}$$

The diameter ratio (d/D) is included in the fully developed expression of the skin friction in order to be consistent with the definition of c_f given in equation (3), in which the skin friction is normalized using the average velocity in the upstream channel. Note also that Reynolds number is the same in both channels ($Q=u_{av}d=U_{av}D$, where U_{av} is the average velocity in the downstream channel). Fig (10) provides some interesting remarks. Skin friction values, in the re-developing region, clearly deviate from their fully developed value. Also, the asymmetric solutions result in higher skin friction values when compared to those of the unstable symmetric solution. Moreover, skin friction values for parabolic inflows, in the re-developing region, are higher than those for uniform inflows.

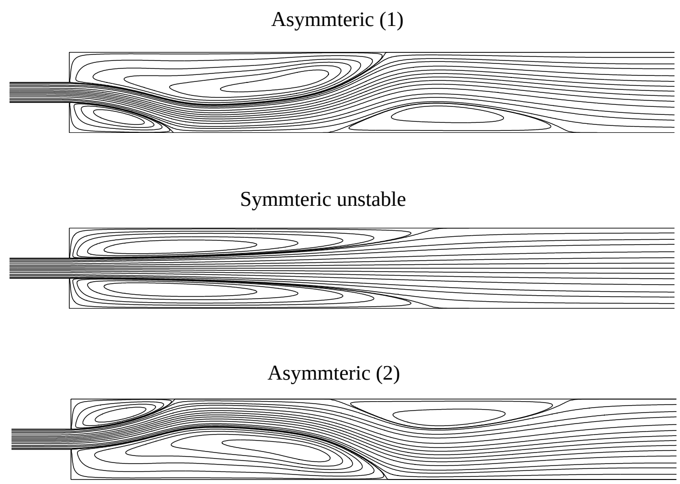

Fig.8- Multiple solutions for Re=100 (Parabolic Inflow)

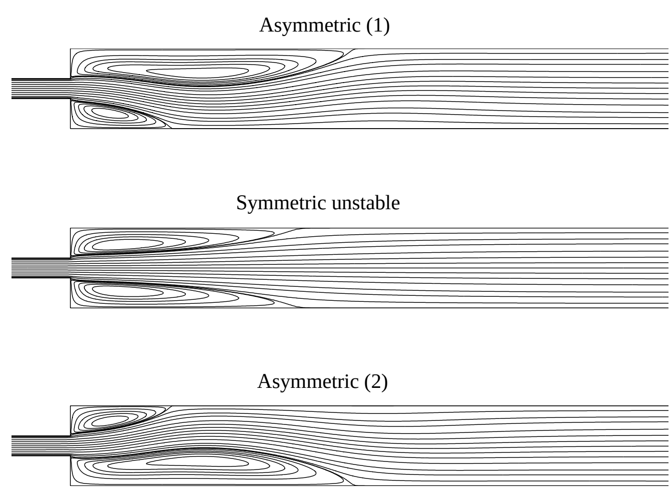

Fig.9- Multiple solutions for Re=100 (Uniform Inflow)

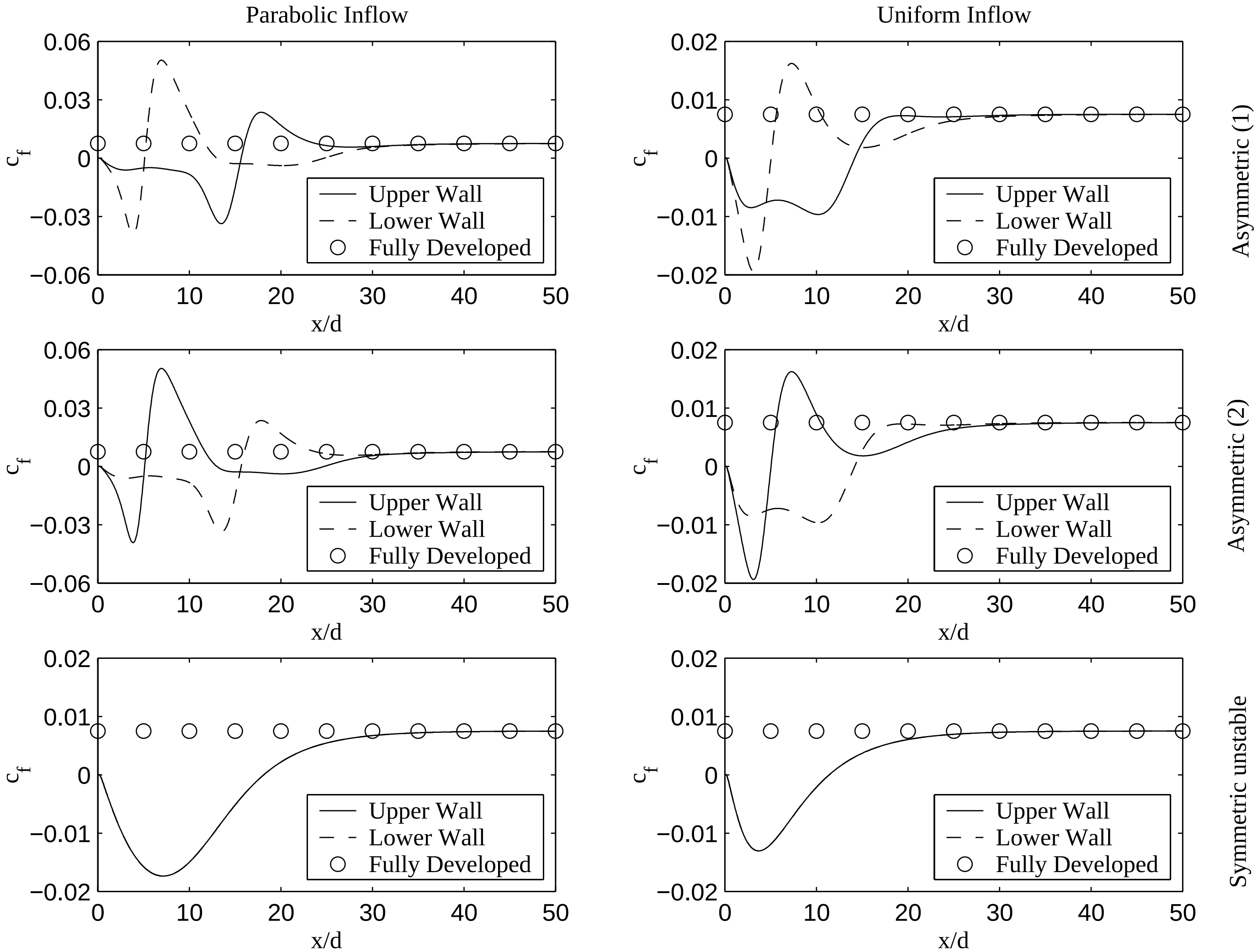

Fig.10- Skin friction distributions for multiple solutions at (Re=100)

In conclusion, it should be noted that the above multiple results clearly demonstrate that certain schemes may not retain symmetry on the discrete level and, therefore, this may have an effect on the numerical simulation and calculation of flow bifurcations. Hence, extreme care should be taken when selecting an iterative solver for fluid flow bifurcation calculations. In this study, although only two solvers are used to demonstrate this effect, however, there is a broad list of methods (and their variants) that could be considered, see for example [27].

4. Conclusions

In the present study, non-uniqueness of incompressible laminar flow through a plane sudden expansion, of ratio 4:1, is investigated numerically. Below a certain Reynolds number, the flow is symmetric with two re-circulation bubbles of equal sizes. Uniform inflows result in smaller re-circulation bubbles than parabolic inflows. Above the critical Reynolds number, the flow undergoes a symmetry-breaking pitchfork bifurcation and asymmetries appear in the flow field. It is found that the type of inflow determines the value of the critical Reynolds number. Uniform inflows tend to stabilize the symmetric solution by delaying the critical Reynolds number to higher values as compared to parabolic inflows. Similar results are obtained for other expansion ratios, confirming that uniform inflows result in higher critical Reynolds numbers and also revealing that the effect of the type of inflow profile is much more significant in case of low expansion ratios. Also, it is shown that explicit solvers recover the stable asymmetric solution, while implicit solvers can recover both the unstable symmetric solution or the stable asymmetric solution, depending on whether the initial guess is symmetric or not. Hence great care should be taken in the selection of iterative solvers for flow bifurcation calculations, especially with respect to retaining symmetry on the discrete level. Skin friction distributions are also provided, showing that the flow regains symmetry after some distance downstream of the expansion plane.

REFERENCES

[1] J. Steinhoff and A. Jameson, ‘Multiple solutions for the transonic potential flow past an airfoil”, AIAA J., 20, pp. 1521-1525, 1982

[2] M. Hafez and A. Dimanlig, “Simulations of compressible inviscid flows over stationary and rotating cylinders”, Acta Mech [Suppl] 4, pp. 241-249, 1994

[3] M. Hafez and A. Dimanlig, “Some anomalies of the numerical solutions of the Euler equations”, Acta Mech 119, pp. 131-140, 1996

[4] C.J. Lee and H.K. Cheng, “An airfoil theory of bifurcating laminar separation from thin obstacles”, J. Fluid Mech., 216, pp. 255-284, 1990

[5] M. Hafez and W. Guo, “Some anomalies of numerical simulation of shock waves. Part II: effect of artificial and real viscosity”, Comp. Fluids, 28, pp. 721-739, 1999

[6] F. Durst, A. Melling, and J.H. Whitelaw, “Low Reynolds number flow over a plane symmetric sudden expansion”, J. Fluid Mech., 64, pp. 111-128, 1974

[7] W. Cherdron, F. Durst, and J.H. Whitelaw, “Asymmetric flows and instabilities in symmetric ducts with sudden expansions”, J. Fluid Mech., 84, pp.13-31, 1978

[8] Y. Ouwa, M. Watanabe, and H. Asawo, “Flow visualization of a two-dimensional water jet in a rectangular channel”, Japanese Journal of Applied Physics, 20, pp. 243-247, 1981

[9] R.M. Fearn, T. Mullin, and K.A. Cliffe, “Nonlinear flow phenomena in a symmetric sudden expansion”, J. Fluid Mech., 211, pp. 595-608, 1990

[10] F. Durst, J.C.F. Pereira and K.A. Cliffe, “The plane symmetric sudden expansion flow at low Reynolds number”, J. Fluid Mech., 248, pp. 567, 1993

[11] M. Shapira, D. Degani, and D. Weihs, “Stability and existence of multiple solutions for viscous flow in suddenly enlarged channels”, Comp. Fluids, 18, pp. 239-258, 1990

[12] Z. Rusak and T. Hawa, “A weakly nonlinear analysis of the dynamics of a viscous flow in a symmetric channel with a sudden expansion”, Phys. Fluids 11, pp. 3629, 1999

[13] T. Hawa and Z. Rusak, “Viscous flow in a slight asymmetric channel with a sudden expansion”, Phys. Fluids 12, pp. 2257, 2000

[14] J. Mizushima and Y. Shiotani, “Structural instability of the bifurcation diagram for two-dimensional flow in a channel with a sudden expansion”, J. Fluid Mech. 420, pp. 131, 2000

[15] N. Allerborn, K. Nandakumar, H. Raszillier, and F. Durst, “Further contributions on the two-dimensional flow in a sudden expansion”, J. Fluid Mech. 330, pp. 169, 1997

[16] F. Battaglia, S. J. Tavener, A. K. Kulkarni, and C. L. Merkle, “Bifurcation of low Reynolds number flows in symmetric channels”, AIAA J. 35, pp. 99, 1997

[17] D. Drikakis, “Bifurcation phenomena in incompressible sudden expansion flows”, Phys. Fluids 9, pp. 76, 1997

[18] A. Revuelta, “On the two-dimensional flow in a sudden expansion with large expansion ratios”, Phys. Fluids, 17 : Art. No. 028102, 2005

[19] D. E. Abbott and S. J. Kline, “Experimental investigation of subsonic turbulent flow over single and double backward facing steps”, J. Basic Eng. Trans. ASME, 84, pp. 317, 1962

[20] P. Neofytou and D. Drikakis, “Non-Newtonian Flow Instability in a Channel with a Sudden Expansion”, J. Non-Newtonian Fluid Mech., 111, 2-3, pp. 127-150, 2003

[21] R. Manica and A. L. De Bortoli, “Simulation of sudden expansion flows for power-law fluids”, J. Non-Newtonian Fluid Mech., 121, pp. 35-40, 2004

[22] I. J. Sobey and T. Mullin, “Calculation of multiple solutions for the two-dimensional Navier–Stokes equations”, Proceedings of Numerical Methods in Fluid Dynamics, edited by M.J. Baines and K.W. Morton, Oxford University Press, Oxford, 4, pp. 417, 1993

[23] P.J. Oliveira and F.T. Pinho, “Pressure drop coefficient of laminar Newtonian flow in axisymmetric sudden expansions”, Int. J. Heat and Fluid Flow, 18, pp. 518-529, 1997

[24] P.S. Scott, F.A. Mirza and J. Vlachopoulos, “A finite element analysis of laminar flows through planar and axisymmteric abrupt expansions”, Comp. Fluids, 14, pp. 423-432, 1986

[25] H.K. Moffat, “Viscous and resistive eddies near a sharp corner”, J. Fluid Mech., 18, pp. 1-18, 1964

[26] T. Hawa and Z. Rusak, “The dynamics of a laminar flow in a symmetric channel with a sudden expansion”, J. Fluid Mech., 436, pp. 283-320, 2001

[27] D. Drikakis and W. Rider, “High-Resolution Methods for Incompressible and Low-Speed Flows”, Springer-Verlag, 2004