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HEAT BALANCE INTEGRAL METHOD APPLIED TO ABLATION PROBLEMS WITH MUSHY ZONES

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ABSTRACT

In the present paper, a new version of heat balance integral method (Goodman's method) is developed to solve the ablation problem with mushy zone. In the classical Goodman's method, three non-linear ordinary differential equations are obtained at the end of the solution procedure. Solving these three non-linear ordinary differential equations lead to the unknowns in the overall problem. In the present paper, existence of the mushy zone is taken into consideration; therefore, the number of unknowns will increase. A proposed approximate method was developed to solve this type of practical importance. The method starts by assuming temperature profile for both liquid and solid phases in such away that some of the boundary conditions will be satisfied, then by some mathematical manipulation two equations in two unknowns that already in the assumed profiles. These unknowns are easily obtained iteratively at each time step.

KEYWORDS:

Phase change problems, Moving boundary problems, Vaporization problems, Mushy zone models.

INTRODUCTION

Moving boundary problems **[1]** are found in a wide range of engineering and industrial applications. Moving boundaries determination is very important and considered as a major part of the solution. The difficulty in solving moving boundary problems refers mainly to the non-linearity caused by the presence of nonlinear boundary condition at the moving boundary. This difficulty makes it difficult in most cases to find an analytical solution, except for very simple problems as in [**2-3**]. Ablation problems are another type of such problems and can be found in different engineering and industrial applications such as material treatment and processing, thermal protection of re-entry vehicles in space technology, laser drilling and cutting in manufacturing, among others [**4**].

There is a major difference between Stefan and ablation problem is that the overall domain in Stefan problem remains fixed in space while the domain in the ablation problem is variable and diminishes in size with time [**5**]. Vaporization is another type of Stefan problems, in such a case, three phases are appearing throughout the process under the assumption of sharp interface between phases. In case of existence of mushy zone between solid and liquid, the situation is then changed completely as in [**6**].

Very few analytical solutions are available and only for simple cases due to the high non-linearity at the moving boundaries [**7**]. Approximate methods such as heat balance integral method [**8**], the moment integral method [**9**] and other methods [**10-12**]. In recent years, efforts concentrate on the use of numerical techniques due to rapid development in computer technology and its high performance. Boundary element method becomes one of the most popular numerical methods, which applied to a wide range of engineering and industrial applications [**13**].

In the present paper, a new version of heat balance integral method (Goodman's method) is developed to solve the ablation problem with mushy zone. In classical Goodman's method, three non-linear ordinary differential equations are obtained at the end of the solution procedure. Solving these three non-linear ordinary equations lead to the unknowns in the overall problem. In the present paper, existence of the mushy zone is taken into consideration; therefore, the number of unknowns will increase. A proposed approximate method was developed to solve this type of practical importance. The method starts by assuming temperature profile for liquid and solid phases in such away that some of the boundary conditions will be satisfied, then by some mathematical manipulation two equations in two unknowns that already in the assumed profiles. These unknowns are easily obtained iteratively at each time step.

2. PROBLEM DESCRIPTION AND FORMULATION

Physical description

A semi-infinite solid initially at a uniform temperature, the boundary $x = 0$ subjected to a high input heat flux therefore three phases occurs. In the present paper, we mainly consider the third stage of this problem in which the vapor is removed upon formation, therefore, the problem remain two phase problem with mushy zone between liquid and solid.

Mathematical Formulation

$$\alpha_\ell \frac{\partial^2 u_\ell}{\partial x^2} = \frac{\partial u_\ell}{\partial t} \qquad R_2(t) < x < R_1(t) \tag{1}$$

$$\alpha_S \frac{\partial^2 u_S}{\partial x^2} = \frac{\partial u_S}{\partial t} \qquad r(t) < x < \ell \tag{2}$$

$$u_S(\ell, t) = U_i \tag{3}$$

$$u_\ell(R_2(t), t) = U_V \tag{4}$$

$$\frac{\partial u_\ell}{\partial x} = \frac{\gamma}{[r(t) - R_1(t)]} \tag{5}$$

$$\left(K_\ell \frac{\partial u_\ell}{\partial x} - K_S \frac{\partial u_S}{\partial x} \right) = \rho L \left[\varepsilon \frac{dR_1(t)}{dt} + (1-\varepsilon) \frac{dr(t)}{dt} \right] \tag{6}$$

In this formulation, we have five unknowns, the temperature distribution in the liquid and solid phases and three moving boundaries, $R_2(t), R_1(t)$ and $r(t)$.

In S. G. Ahmed's paper [14], a tabulated relation between the two parameters, γ and ε was found and in the present paper, we will use these data as input data file when computation was carried out.

3. DERIVATION OF THE PROPOSED METHOD

__Step: 3.1__ The first step in classical Goodman's method is to assume temperature profiles for the phases that appear throughout the whole process:

$$u_\ell(x,t) = \left[\left(\frac{U_m}{R_1 - R_2} \right) + B_2 (x - R_1)^2 \right] (x - R_2) + \left[\left(\frac{U_v}{R_2 - R_1} \right) + B_2 (x - R_2)^2 \right] (x - R_1) \tag{7}$$

$$u_S(x,t) = \left[\frac{U_m}{r - \ell} + B_3 (x - r)^2 \right] (x - \ell) + \left[\frac{U_i}{r - \ell} + B_3 (x - \ell)^2 \right] (r - x) \tag{8}$$

__Step: 3.2__ Integrate equation (1) w. r. t. the space variable x from $x = R_2$ to $x = R_1$, and equation (2) w. r. t. the space variable x from $x = r$ to $x = \ell$ yields:

$$\alpha_\ell \left\{ \left(\frac{\partial u_\ell}{\partial x} \right)_{x=R_1} - \left(\frac{\partial u_\ell}{\partial x} \right)_{x=R_2} \right\} = \frac{d}{dt} \left\{ \int_{x=R_2}^{x=R_1} u_\ell(x,t) dx \right\} + u_\ell(R_1, t) \frac{dR_1}{dt} - u_\ell(R_2, t) \frac{dR_2}{dt} \tag{9}$$

$$\alpha_s \left\{ \left(\frac{\partial u_s}{\partial x} \right)_{x=\ell} - \left(\frac{\partial u_s}{\partial x} \right)_{x=r} \right\} = \frac{d}{dt} \left\{ \int_{x=r}^{x=\ell} u_s(x,t) dx \right\} - u_s(r,t) \frac{dr}{dt} \tag{10}$$

__Step: 3.3__ To find relation between moving boundary velocities and potential derivative. It is known that the total derivative of temperature at the moving boundaries is zero;

$$\frac{D}{Dt} u_\ell(R_2, t) = 0 \tag{11}$$

Equation (11) can be written in an expanded form as follow:

$$\frac{\partial u_\ell(R_2, t)}{\partial x} \frac{dR_2}{dt} + \frac{\partial u_\ell(R_2, t)}{\partial t} = 0 \tag{12}$$

Therefore; the velocities at r and R_1 can be written as follow:

$$\left(\frac{dR_2}{dt} \right)_{x=R_2} = -\alpha_\ell \left(\frac{\partial^2 u_\ell / \partial x^2}{\partial u_\ell / \partial x} \right)_{x=R_2} \tag{13}$$

$$\left(\frac{dR_1}{dt} \right)_{x=R_1} = -\alpha_\ell \left(\frac{\partial^2 u_\ell / \partial x^2}{\partial u_\ell / \partial x} \right)_{x=R_1} \tag{14}$$

Similarly for solid phase;

$$\left(\frac{dr}{dt} \right)_{x=r} = -\alpha_s \left(\frac{\partial^2 u_s / \partial x^2}{\partial u_s / \partial x} \right)_{x=r} \tag{15}$$

__Step: 3.4__ Analytical treatment of equation (9), this step started by differentiating assumed potential profile for liquid phase, then, substituting into the right hand side, one can ensure that this side will equal to zero. The first term in the right hand side is integrated first based on the assumed profile, then differentiating the result w. r. t. time, meanwhile, make use of equations (13) and (14), then after long mathematical manipulation, one can obtain the following equation:

$$\left(\frac{U_m+U_v}{2}\right)\left[\begin{array}{c}\dfrac{-6\alpha_\ell B_2(R_1-R_2)}{\left(\dfrac{U_m-U_v}{R_1-R_2}\right)+B_2(R_1-R_2)^2}\\ +\dfrac{6\alpha_\ell B_2(R_2-R_1)}{\left(\dfrac{U_m-U_v}{R_1-R_2}\right)+B_2(R_2-R_1)^2}\end{array}\right]= \tag{16}$$

$$\frac{\eta_1(t)B_2}{\eta_2(t)+\eta_3(t)B_2}+\frac{\varsigma_1(t)B_2}{\eta_2(t)+\varsigma_2(t)B_2}$$

Where

$$\eta_1(t)=-6\alpha_\ell U_v(R_2-R_1)$$

$$\eta_2(t)=\left(\frac{U_v-U_m}{R_2-R_1}\right)$$

$$\eta_3(t)=(R_2-R_1)^2$$

$$\varsigma_1(t)=-6\alpha_\ell U_m(R_2-R_1)$$

$$\varsigma_2(t)=(R_1-R_2)^2$$

Step: 3.5 Similar procedure is carried out again but on equation (10), the result for this step is as follow:

$$\left[(\ell-r)^2+(r-\ell)^2\right]\left[\left(\frac{U_m-U_i}{r-\ell}\right)-B_3(r-\ell)\right]=$$

$$2(\ell-r)\left[\begin{array}{l}\dfrac{5}{3}B_3r^3-5B_3\ell r^2+5B_3\ell^2r\\ +\dfrac{1}{3}B_3\ell^2-\dfrac{1}{2}U_m+U_i\end{array}\right] \tag{17}$$

It is clear from equation (16), that it contains three unknowns one of them is the unknown function in the assumed profile, given by equation (7). Meanwhile, equation (17) contains two unknowns, one of them is the unknown function in the assumed profile given by equation (8), therefore, the procedure for solution will be illustrated in the next section.

SOLUTION PROCEDURE

(1) Specify initial input data for both mushy zone parameters, ε and γ

(2) Assume initial position for both R_1 and r

(3) From equation (16) evaluate B_2

(4) Evaluate the error in equation (5) making use of eq. (7)

(5) Repeat steps (1) to (4) till satisfying step (4) with prescribed tolerance

(6) From equation (17) evaluate B_3 and then the error in equation (6) but taking into consideration the last and accurate value for the first term in the left hand side. If the error is not within a prescribed error, update the moving boundaries location, if it is satisfied, then go to the next time step.

NUMERICAL RESULTS AND DISCUSSION

The present problem is taken from Ahmed, S.G. **[6]**, in which, aluminum occupy semi-infinite domain initially at uniform temperature u_i. An input heat flux is applied at $x=0$; therefore, three phases are appearing. The vapor is removed upon formation and so the problem still two phase with mushy zone separating liquid and solid. The thermo-physical is shown in table (1). Table 1: Thermo-physical Properties

u_i	932 K
u_V	2543 K
L	376560
K	200 W/mK
ρ	2710
C	1200

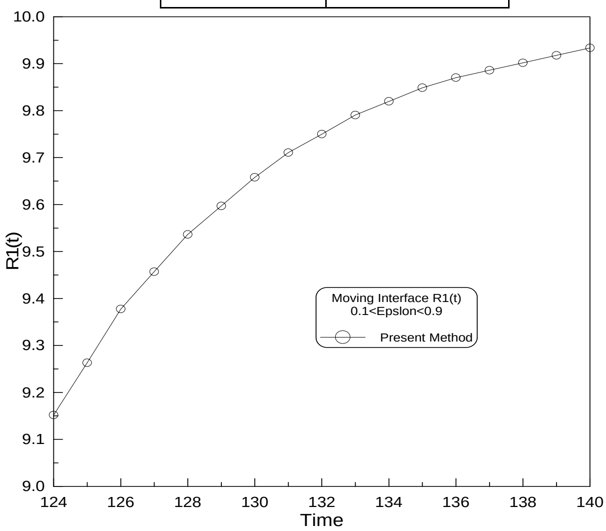

Figure 1: $R_1(t)$ location verses time over $0.1<\varepsilon<0.9$

The results due to the present method appear in the figures (1-4). In figure (1) $R_1(t)$ is plotted against time for the allowed range of $0.1<\varepsilon<0.9$, while in figure (2) $r(t)$ is also plotted against time but at two different values, namely $\varepsilon=0.1$ and $\varepsilon=0.5$.

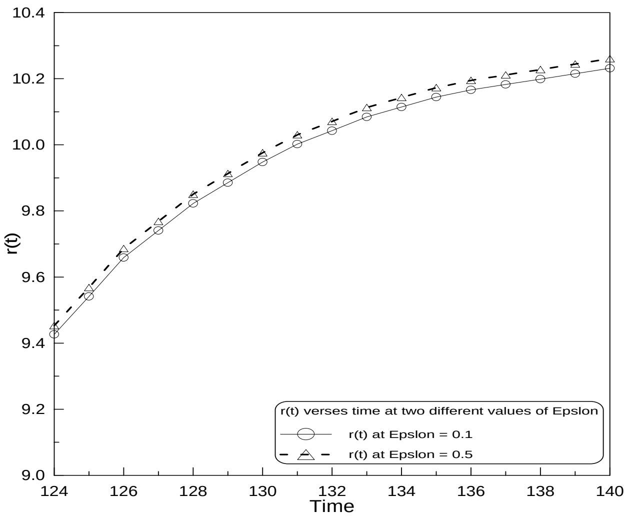

Figure 2: $r(t)$ location verses time at two different ε

CONCLUSION

The heat balance integral method is an old method refers back to Goodman. This method was developed an approximate method for solving classical Stefan melting or solidification problem. In this method the moving interface between solid and was assumed sharp interface, but this is not actual. The present method is a new view of Goodman's method; the new view comes from two ways. The first one is that the moving interface is not sharp but region called mushy region, while second one is the mathematical manipulation, i.e.; in classical Goodman's method the mathematics leads to three nonlinear differential equations, while in the present method we get only two equations, their numerical solution iteratively leads to the complete list of unknowns. The results were good and have many new ideas of mathematical manipulations that will lead to different new approaches of the original methods. Also it should be mentioned that this new version is the first one of many different approaches that deals with mushy zone problems.

NOMENCLATURE

U_i Initial temperature
U_m Melting temperature
U_v Vapor temperature
α Thermal diffusivity
K Conductivity
C Heat capacity
ρ Density
L Latent heat
$R_i(t)$ Moving boundaries for the liquid phase, $i = 1,2$
$r(t)$ Moving boundary separating mushy zone and solid phase

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