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ANALYSIS OF LAMINATED SRM CYLINDRICAL SHELLS UNDER AXIAL THRUST AND PRESSURE LOADS

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ABSTRACT

A closed form analytical solution for laminated cylindrical shell with lateral pressure and axial load is presented. This solution is a continuation to the one presented in reference[6] which includes nonlinear effects due to large deformations and the effects of pre-stress which is encountered with the filament winding of composite solid rocket motor cases. The solution allows the analysis of both symmetric and asymmetric stacking sequences.

Some verification examples are presented for two different stacking sequences; and the effects of motor length, the variation of the axial thrust and the internal pressure, as well as the pre-stress are investigated.

KEYWORDS:
Analysis, Laminatrd, Composite, Shells, Preload, Axial thrust, Pressure

INTRODUCTION

At the date of this writing, composite materials have been used in the aerospace industry for over five decades. One important application of composite materials is in solid rocket motor cases. These structures are characterized by geometries in which the motors have one (or more) axes of symmetry and the thickness of the composite shell can be considered small in comparison to other dimensions. In general, these composite shells are produced by a filament winding process. In this process filament bundles saturated with a matrix resin are wound around a mandrel that defines the finished shape of the SRM. The filament bundles must be held in tension during this process to ensure the finished quality and prevent the slipping of the filament tow. This tension is maintained throughout the winding process and the subsequent cure cycle. As a result, the finished part is preloaded. Most of the analysis techniques for filament wound systems did not take this preload into account and neglect it. The purpose of this paper is to develop a better understanding of the response of filament wound SRM cases and the effect of preload on its response.

THEORITICAL ANALYSIS

The shell to be considered in this development is shown in Figure 1. It is assumed to have an average radius r, length L, and wall thickness h. The coordinate system used in the analysis is also shown in the figure. The displacements of the shell are u, v, and w in the x, ϕ, and z directions, respectively, as shown in the figure.

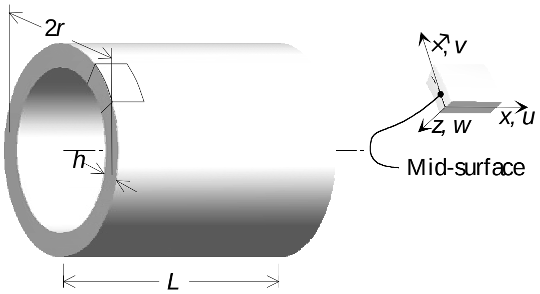

Figure 1. Axisymmetric Shell Dimensions and Associated Coordinate System with Displacement Directions.

Strain-Displacement Relations

The following assumptions are made based on small deflection theory and the application of the Kirchoff-Love hypothesis:
1. The thickness of the shell, h, is very small in comparison with other dimensions, i.e.: $h << r$, and $h << L$.
2. The normal to the reference middle surface before deformation remains normal after deformation.
3. The normal stress through the thickness is zero.

As a consequence of assumption 2 and for a constant z, the in plane shear strains are negligible, i.e.: $\gamma_{xz} = \gamma_{\phi z} \cong 0$. From this result and assumption 3, the normal strain through the thickness is zero, i.e.: $\varepsilon_z = 0$. Based on these assumptions, the displacements can be written as:

$$u = u_0 - z\frac{\partial w_0}{\partial x}\ ,$$
$$v = v_0 - z\frac{\partial w_0}{r\partial\phi}\ , \qquad (1)$$
$$w = w_0\ ,$$

where u_0, v_0, w_0 are the displacements of the middle surface in the x, ϕ, and z directions respectively.

Therefore, from the assumed displacement field and the definition of strain, the strain-displacement relations for any point in the shell are found to be:

$$\varepsilon_x = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2}\ ,$$
$$\varepsilon_\phi = \frac{1}{r}\left(\frac{\partial v_0}{\partial\phi} - w_0\right) - \frac{z}{r^2}\frac{\partial^2 w_0}{\partial\phi^2}\ , \qquad (2)$$
$$\gamma_{x\phi} = \frac{\partial v_0}{\partial x} + \frac{1}{r}\frac{\partial u_0}{\partial\phi} - \frac{2z}{r}\frac{\partial^2 w_0}{\partial x\partial\phi}.$$

The above expressions may be rewritten in terms of the mid-plane strains and curvatures. These expressions are:

$$\varepsilon_x = \varepsilon_x^0 + z\kappa_x^0\ ,$$
$$\varepsilon_\phi = \varepsilon_\phi^0 + z\kappa_\phi^0\ , \qquad (3)$$
$$\gamma_{x\phi} = \gamma_{x\phi}^0 + z\kappa_{x\phi}^0\ ,$$

where $\varepsilon_x^0, \varepsilon_\phi^0, \gamma_{x\phi}^0$ are the mid-plane strains and

$$\kappa_x^0 = -\frac{\partial^2 w_0}{\partial x^2} \ ,$$

$$\kappa_\phi^0 = -\frac{1}{r^2}\frac{\partial^2 w_0}{\partial \phi^2} \ , \qquad (4)$$

$$\kappa_{x\phi}^0 = -\frac{2}{r}\frac{\partial^2 w_0}{\partial x \partial \phi} \ ,$$

are the mid-plane curvatures.

Stress-Strain Relationship

For this development assume that the shell is in a state of plane stress. Therefore, $\sigma_z = \tau_{xz} = \tau_{\phi z} = 0$. This assumption is compatible with the previous assumptions. Further, consider the shell to be made of orthotropic lamina. Therefore, each lamina of the shell has a set of principal material directions. These directions form a local coordinate system in the lamina, i.e.: 1-, 2-, *z* directions. Under the plane stress state, the stress-strain relations for the *k*th lamina in the local lamina coordinate directions are:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}_k = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}_k \ , \qquad (5)$$

where Q_{ij} are the principal stiffnesses of the *k*th lamina, σ_i and τ_{12} are the normal and shear stresses in the principal material directions, respectively, and ε_i and γ_{12} the normal and shear strains in the principal material directions, respectively. The notation used in this development is taken from that used by Jones[1]. These stiffness terms are directly related to the engineering properties (moduli and Poisson's ratio) of the lamina. Expressions for these terms can be found in various texts[1,2,3].

Through the process of coordinate transformation, the stress-strain relations for the *k*th lamina in the global coordinate system of the shell can be shown to be:

$$\begin{Bmatrix} \sigma_x \\ \sigma_\phi \\ \tau_{x\phi} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\phi \\ \gamma_{x\phi} \end{Bmatrix}_k \ , \qquad (6)$$

where $\overline{Q}_{ij}$ are the stiffnesses of the *k*th lamina in the global coordinate system. Expressions for these terms in terms of θ_k, the angle between the first principal material direction and the *x* direction, the winding angle, can be found in various texts[1,2,3]. The stress in the *k*th lamina may then be expressed in terms of the mid-plane strains and curvatures by substituting equation (3) into equation (6). Thus producing the lamina stress-strain/curvature relationship:

$$\begin{Bmatrix} \sigma_x \\ \sigma_\phi \\ \tau_{x\phi} \end{Bmatrix}_k = [\overline{Q}]_k \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_\phi^0 \\ \gamma_{x\phi}^0 \end{Bmatrix} + z[\overline{Q}]_k \begin{Bmatrix} \kappa_x^0 \\ \kappa_\phi^0 \\ \kappa_{x\phi}^0 \end{Bmatrix} . \qquad (7)$$

Force and Moment Resultants

Consider the laminated cylindrical shell to have *n* laminae. Further, let a typical lamina, *k*, be bounded by the surfaces $z = h_{k-1}$ and $z = h_k$. As a consequence of the assumption that $\frac{z}{r} << 1$, the force and moment resultants can be written as:

$$\begin{Bmatrix} N_x^o \\ N_\phi^o \\ N_{x\phi}^o \end{Bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{Bmatrix} \sigma_x \\ \sigma_\phi \\ \tau_{x\phi} \end{Bmatrix}_k dz\,, \tag{8}$$

$$\begin{Bmatrix} M_x^o \\ M_\phi^o \\ M_{x\phi}^o \end{Bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{Bmatrix} \sigma_x \\ \sigma_\phi \\ \tau_{x\phi} \end{Bmatrix}_k zdz\,, \tag{9}$$

where N_x^o and N_ϕ^o are defined as the normal in-plane forces per unit length in the x, and ϕ directions respectively, $N_{x\phi}^o$ is defined as the in-plane shear force per unit length in the x-ϕ plane, M_x^o and M_ϕ^o are defined as the bending moments per unit length in the x, and ϕ directions respectively, and $M_{x\phi}^o$ is defined as the twisting moment per unit length in the x-ϕ plane.

Finally, the relationships between the force and moment resultants and the strains and curvatures of the mid-plane are obtained by substituting the results found in equation (7) into equations (8) and (9). Noting that the mid-plane strains and curvatures, $\{\varepsilon^0\}$ and $\{\kappa^0\}$ are independent of the variable of integration and the stiffnesses, $[\overline{Q}]_k$, are constant for each lamina, this relation becomes:

$$\left\{\frac{N^o}{M^o}\right\} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right] \left\{\frac{\varepsilon^0}{\kappa^0}\right\}, \tag{10}$$

where:

$$A_{ij} = \sum_{k=1}^{n} \left(\overline{Q}_{ij}\right)_k \left(h_{k-1} - h_k\right) \quad i, j = 1,2,6,$$

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{n} \left(\overline{Q}_{ij}\right)_k \left(h_{k-1}^2 - h_k^2\right)$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n} \left(\overline{Q}_{ij}\right)_k \left(h_{k-1}^3 - h_k^3\right) \tag{11}$$

which is as found in classical lamination theory.

PRESTRESS AND THE FORCE AND MOMENT RESULTANTS

During the winding process, the filaments that make up the lamina are wrapped or wound around a mandrel. This process requires a tension be maintained on the filament bundle or tow throughout the winding process. The result is a preload or prestress in each lamina of the shell. The prestress or residual stress remaining in the part after all processing is dependent on a number of factors which include: shell geometry, fiber tension, wrap angle, relative speed between the mandrel and feeder, resin viscosity, cure method, and cure process or cycle. For the purposes of this analysis it is assumed that the post-processing tension in a fiber bundle or tow is constant along the length of the bundle within a given lamina. This tension is defined to be *T*. Therefore, the resulting prestress in the lamina in the local coordinate system is:

$$\begin{Bmatrix} \sigma_1^* \\ \sigma_2^* \\ \tau_{12}^* \end{Bmatrix} = \begin{Bmatrix} \sigma_T \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} T/A_f \\ 0 \\ 0 \end{Bmatrix}, \tag{12}$$

where σ_1^*, σ_2^*, and τ_{12}^* are the components of the prestress in the principal material directions, σ_T is the stress on the fiber bundle, and A_f is the cross-sectional area of the fiber bundle.

Given the wrap angle, defined as θ_k, is constant for the *k*th lamina, the prestress can be transformed into the global coordinate system. The result of this transform is:

$$\begin{Bmatrix} \sigma_x^* \\ \sigma_y^* \\ \tau_{xy}^* \end{Bmatrix} = \sigma_{T_k} \begin{Bmatrix} \cos^2\theta_k \\ \sin^2\theta_k \\ \sin\theta_k \cos\theta_k \end{Bmatrix}, \tag{13}$$

where σ_{T_k} is the prestress in the *k*th lamina. The force and bending moment resultants due to the prestress can be found by substituting equation (13) into equations (8) and (9). The result is:

$$\begin{Bmatrix} N_x^* \\ N_y^* \\ N_{x\phi}^* \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x^* \\ \sigma_\phi^* \\ \tau_{x\phi}^* \end{Bmatrix} dz = \sum_{k=1}^{n} \sigma_{T_k} \begin{Bmatrix} \cos^2\theta \\ \sin^2\theta \\ \cos\theta\sin\theta \end{Bmatrix}_k \left(h_{k-1} - h_k\right) \tag{14}$$

$$\begin{Bmatrix} M_x^* \\ M_\phi^* \\ M_{x\phi}^* \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x^* \\ \sigma_\phi^* \\ \tau_{x\phi}^* \end{Bmatrix} z dz = \frac{1}{2}\sum_{k=1}^{n} \sigma_{T_k} \begin{Bmatrix} \cos^2\theta \\ \sin^2\theta \\ \cos\theta\sin\theta \end{Bmatrix}_k \left(h_{k-1}^2 - h_k^2\right) \tag{15}$$

where * designate the resultants due to the prestress terms. These prestress terms exist in the shell in the undeformed or reference configuration. Therefore, the laminate constitutive relation as given in equation (10) now takes the form:

$$\left\{ \frac{N}{M} \right\} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array} \right] \left\{ \frac{\varepsilon^0}{\kappa^0} \right\} + \left\{ \frac{N^*}{M^*} \right\}. \tag{16}$$

EQUILIBRIUM EQUATIONS FOR LAMINATED CYLINDRICAL SHELLS

Consider an infinitesimal element from a laminated cylindrical shell. This element is cut from the shell by two adjacent axial sections and two adjacent sections perpendicular to the axis of the cylinder, as shown in Figure 1. The corresponding element of the middle surface of the shell is shown in Figure 2 in its deformed configuration. In Figure 3, the force and bending moment resultants are designated as $\overline{N}_i$, $\overline{N}_{ij}$, $\overline{M}_i$, and $\overline{M}_{ij}$, differentiating them from the resultants in the undeformed configuration. From Timeoshenko and Woinowsky-Krieger[4], Timoshenko and Gere[5], and the assumption that $\frac{z}{r} << 1$, then $\overline{N}_{x\phi} \neq \overline{N}_{\phi x}$, $\overline{M}_{x\phi} \neq \overline{M}_{\phi x}$, and the force equilibrium equations are:

$$\frac{\partial \overline{N}_x}{\partial x} + \frac{1}{r}\frac{\partial \overline{N}_{\phi x}}{\partial \phi} - \overline{N}_{xz}\frac{\partial^2 w_0}{\partial x^2} - \overline{N}_{x\phi}\frac{\partial^2 v_0}{\partial x^2} - \frac{\overline{N}_{\phi z}}{r}\left(\frac{\partial v_0}{\partial x} + \frac{\partial^2 w_0}{\partial x \partial \phi}\right) - \frac{\overline{N}_{\phi}}{r}\left(\frac{\partial^2 v_0}{\partial x \partial \phi} - \frac{\partial w_0}{\partial x}\right) + \overline{p}_x = 0, \tag{17}$$

$$\frac{1}{r}\frac{\partial \overline{N}_\phi}{\partial \phi} + \frac{\partial \overline{N}_{x\phi}}{\partial x} + \overline{N}_x\frac{\partial^2 v_0}{\partial x^2} - \frac{\overline{N}_{xz}}{r}\left(\frac{\partial v_0}{\partial x} + \frac{\partial^2 w_0}{\partial x \partial \phi}\right) + \frac{\overline{N}_{\phi x}}{r}\left(\frac{\partial^2 v_0}{\partial x \partial \phi} - \frac{\partial w_0}{\partial x}\right) - \frac{\overline{N}_{\phi z}}{r}\left(1 + \frac{1}{r}\frac{\partial v_0}{\partial \phi} + \frac{1}{r}\frac{\partial^2 w_0}{\partial \phi^2}\right) + \overline{p}_\phi = 0, \tag{18}$$

$$\frac{\partial \overline{N}_{xz}}{\partial x} + \frac{1}{r}\frac{\partial \overline{N}_{\phi z}}{\partial \phi} + \frac{\left(\overline{N}_{x\phi} + \overline{N}_{\phi x}\right)}{r}\left(\frac{\partial v_0}{\partial x} + \frac{\partial^2 w_0}{\partial x \partial \phi}\right) + \overline{N}_x\frac{\partial^2 w_0}{\partial x^2} + \frac{\overline{N}_{\phi}}{r}\left(1 + \frac{1}{r}\frac{\partial v_0}{\partial \phi} + \frac{1}{r}\frac{\partial^2 w_0}{\partial \phi^2}\right) + \overline{p}_z = \rho h\frac{\partial^2 w_0}{\partial t^2}, \tag{19}$$

where both in-plane body force terms, $\rho h\frac{\partial^2 u_0}{\partial t^2}$ and $\rho h\frac{\partial^2 v_0}{\partial t^2}$ are very small compared to the out of plane, z-direction, term. Further, the moment equilibrium equations, neglecting the rotary inertia terms, are:

$$\frac{\partial \overline{M}_{x\phi}}{\partial x}-\frac{1}{r}\frac{\partial \overline{M}_{\phi}}{\partial \phi}-\frac{\overline{M}_{\phi x}}{r}\left(\frac{\partial^2 v_0}{\partial x\partial \phi}-\frac{\partial w_0}{\partial x}\right)+\overline{M}_{x}\frac{\partial^2 v_0}{\partial x^2}+\overline{N}_{\phi z}=0, \tag{20}$$

$$\frac{1}{r}\frac{\partial \overline{M}_{\phi x}}{\partial \phi}+\frac{\partial \overline{M}_{x}}{\partial x}+\overline{M}_{x\phi}\frac{\partial^2 v_0}{\partial x^2}-\frac{\overline{M}_{\phi}}{r}\left(\frac{\partial^2 v_0}{\partial x\partial \phi}-\frac{\partial w_0}{\partial x}\right)+\overline{N}_{xz}=0, \tag{21}$$

$$\frac{\overline{M}_{x}}{r}\left(\frac{\partial v_0}{\partial x}+\frac{\partial^2 w_0}{\partial x\partial \phi}\right)+\overline{M}_{\phi x}\left(\frac{1}{r}+\frac{1}{r^2}\frac{\partial v_0}{\partial \phi}+\frac{1}{r^2}\frac{\partial^2 w_0}{\partial \phi^2}\right)+\overline{M}_{x\phi}\frac{\partial^2 w_0}{\partial x^2}-\frac{\overline{M}_{\phi}}{r}\left(\frac{\partial v_0}{\partial x}+\frac{\partial^2 w_0}{\partial x\partial \phi}\right) \tag{22}$$
$$+\overline{N}_{x\phi}-\overline{N}_{\phi x}=0.$$

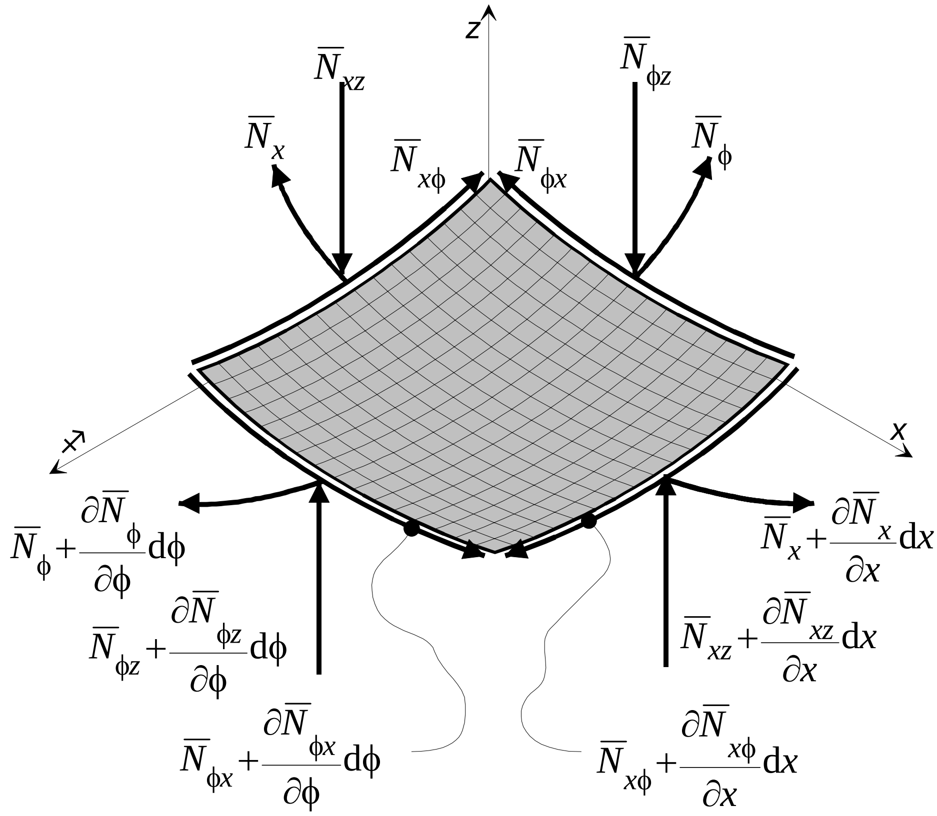

Figure 2. Force Resultants on the Midplane of an Infinitesmal Shell Element in the Deformed Configuration.

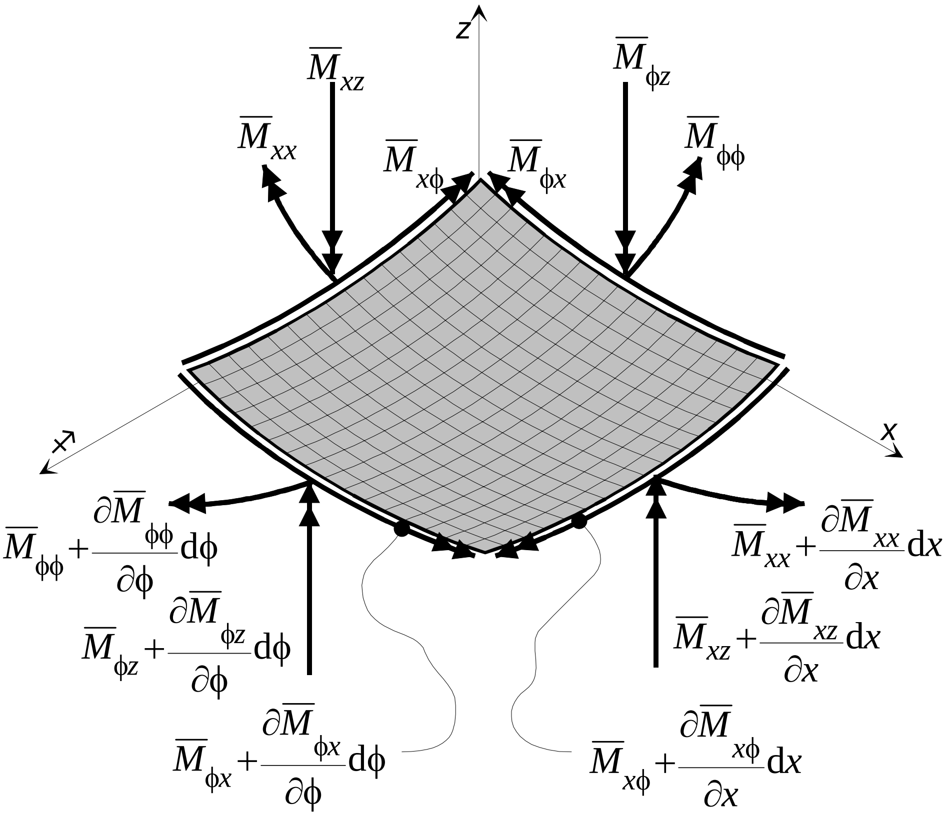

Figure 3. Moment Resultants on the Midplane of an Infinitesmal Shell Element in the Deformed Configuration.

Static Response of Laminated Cylindrical Shells Under Uniform Lateral and Axial Compression

The set of differential equations (17) through (22) represent the equilibrium of cylindrical shells in terms of the internal resultant forces and moments in the deformed configuration, and the displacements of the middle surface. To obtain the displacement field equations (16) through (22) must be solved simultaneously. Alternately, the force / moment -- displacement / curvature relations may be substituted into equations (17) through (22) to obtain a set of simultaneous partial differential equations in the displacements of the mid-plane. This development uses the later approach.

Consider at this point a special case: static response to uniform lateral compression (internal or external pressure on the shell in the radial, i.e.: z-direction) and uniform axial compression. The assumption of static response simplifies the z-direction force equilibrium equation, causing the inertia term to be zero. The lateral compressive pressure is $p_z = p$ and the applied axial compressive load per unit length is $N_x = P$. These loads are constant, therefore, the resultant shear forces are very small compared to the other forces. Consequently, the products of these forces with displacement terms and derivatives of displacement are negligible. Hence, the force balance equations, equations (17) through (19), take the following form:

$$\frac{\partial \overline{N}_x}{\partial x}+\frac{1}{r}\frac{\partial \overline{N}_{\phi x}}{\partial \phi}-\overline{N}_{x\phi}\frac{\partial^2 v_0}{\partial x^2}-\frac{\overline{N}_\phi}{r}\left[\frac{\partial^2 v_0}{\partial x\partial\phi}-\frac{\partial w_0}{\partial x}\right]=0, \tag{23}$$

$$\frac{1}{r}\frac{\partial \overline{N}_\phi}{\partial \phi}+\frac{\partial \overline{N}_{x\phi}}{\partial x}+\overline{N}_x\frac{\partial^2 v_0}{\partial x^2}+\frac{\overline{N}_{\phi x}}{r}\left[\frac{\partial^2 v_0}{\partial x\partial\phi}-\frac{\partial w_0}{\partial x}\right]-\frac{\overline{N}_{\phi z}}{r}=0, \tag{24}$$

$$\frac{\partial \overline{N}_{xz}}{\partial x}+\frac{1}{r}\frac{\partial \overline{N}_{\phi z}}{\partial \phi}+\frac{\overline{N}_{x\phi}+\overline{N}_{\phi x}}{r}\left(\frac{\partial v_0}{\partial x}+\frac{\partial^2 w_0}{\partial x\partial\phi}\right)+\overline{N}_x\frac{\partial^2 w_0}{\partial x^2}+\frac{\overline{N}_\phi}{r}\left(1+\frac{1}{r}\frac{\partial v_0}{\partial\phi}+\frac{1}{r}\frac{\partial^2 w_0}{\partial\phi^2}\right)+\overline{p}=0. \tag{25}$$

By substituting the moment equilibrium equations, equations (20) and (21) into the ϕ- and z-direction force equilibrium equations, equations (24) and (25) the governing equations for this case reduce to:

$$\frac{\partial \overline{N}_x}{\partial x}+\frac{1}{r}\frac{\partial \overline{N}_{\phi x}}{\partial \phi}-\overline{N}_{x\phi}\frac{\partial^2 v_0}{\partial x^2}-\frac{\overline{N}_\phi}{r}\left[\frac{\partial^2 v_0}{\partial x\partial\phi}-\frac{\partial w_0}{\partial x}\right]=0, \tag{26}$$

$$\begin{aligned}&\frac{1}{r}\frac{\partial \overline{N}_\phi}{\partial \phi}+\frac{\partial \overline{N}_{x\phi}}{\partial x}+\overline{N}_x\frac{\partial^2 v_0}{\partial x^2}+\frac{\overline{N}_{\phi x}}{r}\left[\frac{\partial^2 v_0}{\partial x\partial\phi}-\frac{\partial w_0}{\partial x}\right]\\&\quad+\frac{1}{r}\left[\frac{\partial \overline{M}_{x\phi}}{\partial x}-\frac{1}{r}\frac{\partial \overline{M}_\phi}{\partial\phi}-\frac{\overline{M}_{\phi x}}{r}\left[\frac{\partial^2 v_0}{\partial x\partial\phi}-\frac{\partial w_0}{\partial x}\right]+\overline{M}_x\frac{\partial^2 v_0}{\partial x^2}\right]=0,\end{aligned} \tag{27}$$

$$\begin{aligned}&\frac{\partial^2 \overline{M}_x}{\partial x^2}+\frac{1}{r}\left[\frac{\partial^2 \overline{M}_{\phi x}}{\partial x\partial\phi}-\frac{\partial^2 \overline{M}_{x\phi}}{\partial x\partial\phi}\right]+\frac{1}{r^2}\frac{\partial^2 \overline{M}_\phi}{\partial\phi^2}+\left[\frac{\partial \overline{M}_{x\phi}}{\partial x}+\frac{1}{r}\frac{\partial \overline{M}_x}{\partial\phi}\right]\frac{\partial^2 v_0}{\partial x^2}\\&\quad-\frac{1}{r}\left[\frac{\partial \overline{M}_\phi}{\partial x}-\frac{1}{r}\frac{\partial \overline{M}_{\phi x}}{\partial\phi}\right]\left[\frac{\partial^2 v_0}{\partial x\partial\phi}-\frac{\partial w_0}{\partial x}\right]+\frac{\overline{M}_x}{r}\frac{\partial^3 v_0}{\partial x^2\partial\phi}+\overline{M}_{x\phi}\frac{\partial^3 v_0}{\partial x^3}\\&\quad+\frac{\overline{M}_{\phi x}}{r^2}\left[\frac{\partial^3 v_0}{\partial x\partial\phi^2}-\frac{\partial^2 w_0}{\partial x\partial\phi}\right]+\frac{\overline{M}_\phi}{r}\left[\frac{\partial^3 v_0}{\partial x^2\partial\phi}-\frac{\partial^2 w_0}{\partial x^2}\right]\\&\quad+\frac{\overline{N}_{\phi x}+\overline{N}_{x\phi}}{r}\left[\frac{\partial v_0}{\partial x}+\frac{\partial^2 w_0}{\partial x\partial\phi}\right]+\overline{N}_x\frac{\partial^2 w_0}{\partial x^2}+\frac{\overline{N}_\phi}{r}\left[1+\frac{1}{r}\frac{\partial v_0}{\partial\phi}+\frac{1}{r}\frac{\partial^2 w_0}{\partial\phi^2}\right]+\overline{p}=0.\end{aligned} \tag{28}$$

$$\begin{aligned}&\frac{\overline{M}_x}{r}\left(\frac{\partial v_0}{\partial x}+\frac{\partial^2 w_0}{\partial x\partial\phi}\right)+\overline{M}_{\phi x}\left(\frac{1}{r}+\frac{1}{r^2}\frac{\partial v_0}{\partial\phi}+\frac{1}{r^2}\frac{\partial^2 w_0}{\partial\phi^2}\right)+\overline{M}_{x\phi}\frac{\partial^2 w_0}{\partial x^2}-\frac{\overline{M}_\phi}{r}\left(\frac{\partial v_0}{\partial x}+\frac{\partial^2 w_0}{\partial x\partial\phi}\right)\\&\quad+\overline{N}_{x\phi}-\overline{N}_{\phi x}=0.\end{aligned} \tag{29}$$

Equations (26) through (29) are the complete equilibrium equations for the static response of a laminated cylindrical shell under uniform lateral and axial compression.

After the manner of Timoshenko[5] the stretching of the middle surface of the shell can be taken into account by defining the following relations:

$$\bar{N}_x = \left(N_x - P\right)\left(1+\varepsilon_\phi^0\right)$$
$$\bar{N}_\phi = \left(N_\phi - pr\right)\left(1+\varepsilon_x^0\right)$$
$$\bar{N}_{x\phi} = N_{x\phi}\left(1+\varepsilon_\phi^0\right)$$
$$\bar{N}_{\phi x} = N_{x\phi}\left(1+\varepsilon_x^0\right)$$
$$\bar{M}_x = M_x\left(1+\varepsilon_\phi^0\right)$$
$$\bar{M}_\phi = M_\phi\left(1+\varepsilon_x^0\right)$$
$$\bar{M}_{x\phi} = M_{x\phi}\left(1+\varepsilon_\phi^0\right)$$
$$\bar{M}_{\phi x} = M_{x\phi}\left(1+\varepsilon_x^0\right)$$
$$\bar{p} = p\left(1+\varepsilon_x^0\right)\left(1+\varepsilon_\phi^0\right) \qquad (30)$$

where:

$$N_x = N_x^o + N_x^*,$$
$$N_\phi = N_\phi^o + N_\phi^*,$$
$$N_{x\phi} = N_{x\phi}^o + N_{x\phi}^*,$$
$$M_x = M_x^o + M_x^*,$$
$$M_\phi = M_\phi^o + M_\phi^*,$$
$$M_{x\phi} = M_{x\phi}^o + M_{x\phi}^*.$$

Recall that N_i^o and M_i^o are the resultant forces and moments in the undeformed geometry, respectively, and N_i^* and M_i^* are the prestress resultant forces and moments in the undeformed geometry, respectively. The governing equations for equilibrium of the shell in terms of the variables in the undeformed configuration are found by substituting the definitions from equation (30) into equations (26) through (29). After rearranging these equations and noting the assumptions on laminate configuration are such that the resultant forces and moments due to the prestress are independent of x and ϕ, the following equations are obtained:

$$
\begin{aligned}
&\frac{\partial N_x^o}{\partial x}\left(1+\frac{1}{r}\left(\frac{\partial v_0}{\partial \phi}-\frac{w_0}{r}\right)\right)+\frac{1}{r}\frac{\partial N_{x\phi}^o}{\partial \phi}\left(1+\frac{\partial u_0}{\partial x}\right)\\
&\quad+\left[p+\frac{\left(N_x^o-N_\phi^o\right)+\left(N_x^*-N_\phi^*\right)-P}{r}+\frac{\partial u_0}{\partial x}\left(p-\frac{N_\phi^o+N_\phi^*}{r}\right)\right]\left(\frac{\partial^2 v_0}{\partial x\partial \phi}-\frac{\partial w_0}{\partial x}\right)\\
&\quad+\left(\frac{N_{x\phi}^o+N_{x\phi}^*}{r}\right)\left[\frac{\partial^2 u_0}{\partial x\partial \phi}-\left(r+\frac{\partial v_0}{\partial \phi}-w_0\right)\frac{\partial^2 v_0}{\partial x^2}\right]=0,
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
&\frac{1}{r}\left(\frac{\partial N_\phi^o}{\partial \phi}-\frac{1}{r}\frac{\partial M_\phi^o}{\partial \phi}\right)\left(1+\frac{\partial u_0}{\partial x}\right)+\left(\frac{\partial N_{x\phi}^o}{\partial x}-\frac{1}{r}\frac{\partial M_{x\phi}^o}{\partial x}\right)\left(1+\frac{1}{r}\left(\frac{\partial v_0}{\partial \phi}-\frac{w_0}{r}\right)\right)\\
&\quad+\left(\frac{N_\phi^o+N_\phi^*}{r}-\frac{M_\phi^o+M_\phi^*}{r^2}-p\right)\frac{\partial^2 u_0}{\partial x\partial \phi}+\frac{\left(N_{x\phi}^o+N_{x\phi}^*\right)}{r}\left(\frac{\partial^2 v_0}{\partial x\partial \phi}-\frac{\partial w_0}{\partial x}\right)\left(2+\frac{\partial u_0}{\partial x}\right)\\
&\quad+\left(N_x^o+N_x^*-\frac{M_x^o+M_x^*}{r}-P\right)\frac{\partial^2 v_0}{\partial x^2}\left(1+\frac{1}{r}\left(\frac{\partial v_0}{\partial \phi}-\frac{w_0}{r}\right)\right)\\
&\quad-\frac{M_\phi^o+M_\phi^*}{r^2}\frac{\partial u_0}{\partial x}\left(\frac{\partial^2 v_0}{\partial x\partial \phi}-\frac{\partial w_0}{\partial x}\right)=0,
\end{aligned}
\tag{32}
$$

$$\frac{\partial^2 M_x^o}{\partial x^2}\left(1+\frac{1}{r}\frac{\partial v_0}{\partial\phi}-\frac{w_0}{r}\right)+\frac{1}{r^2}\frac{\partial^2 M_\phi^o}{\partial\phi^2}\left(1+\frac{\partial u_0}{\partial x}\right)+\frac{1}{r}\frac{\partial^2 M_{x\phi}^o}{\partial x\partial\phi}\left(\frac{\partial u_0}{\partial x}+\frac{w_0}{r}-\frac{1}{r}\frac{\partial v_0}{\partial\phi}\right)$$
$$+\frac{2}{r}\frac{\partial M_x^o}{\partial x}\left(\frac{\partial^2 v_0}{\partial x\partial\phi}-\frac{\partial w_0}{\partial x}\right)+\frac{1}{r}\frac{\partial M_x^o}{\partial\phi}\left(\frac{\partial^2 v_0}{\partial x^2}\left(1+\frac{1}{r}\frac{\partial v_0}{\partial\phi}-\frac{w_0}{r}\right)\right)+\frac{2}{r^2}\frac{\partial M_\phi^o}{\partial\phi}$$
$$+\frac{1}{r}\frac{\partial M_\phi^o}{\partial x}\left(\left(\frac{\partial w_0}{\partial x}-\frac{\partial^2 v_0}{\partial x\partial\phi}\right)\left(1+\frac{\partial u_0}{\partial x}\right)\right)$$
$$+\frac{\partial M_{x\phi}^o}{\partial x}\left(\left(\frac{1}{r}\frac{\partial^2 u_0}{\partial x\partial\phi}-\frac{1}{r}\frac{\partial^2 v_0}{\partial\phi^2}+\frac{1}{r}\frac{\partial w_0}{\partial\phi}\right)+\frac{\partial^2 v_0}{\partial x^2}\left(1+\frac{w_0}{r}-\frac{1}{r}\frac{\partial v_0}{\partial\phi}\right)\right)$$
$$+\frac{\partial M_{x\phi}^o}{\partial\phi}\left(\left(\frac{1}{r}\frac{\partial^2 u_0}{\partial x^2}-\frac{1}{r}\frac{\partial w_0}{\partial x}-\frac{1}{r}\frac{\partial^2 v_0}{\partial x\partial\phi}\right)+\frac{1}{r^2}\left(\frac{\partial^2 v_0}{\partial x\partial\phi}-\frac{\partial w_0}{\partial x}\right)\left(1+\frac{\partial u_0}{\partial x}\right)\right)$$
$$+\frac{M_x^o+M_x^*}{r}\left(\left(2+\frac{1}{r}\frac{\partial v_0}{\partial\phi}-\frac{w_0}{r}\right)\frac{\partial^3 v_0}{\partial x^2\partial\phi}+\frac{1}{r}\left(\frac{\partial^2 v_0}{\partial\phi^2}-\frac{\partial w_0}{\partial\phi}\right)\frac{\partial^2 v_0}{\partial x^2}-\frac{\partial^2 w_0}{\partial x^2}\right)$$
$$+\frac{M_\phi^o+M_\phi^*}{r}\left(\frac{1}{r}\frac{\partial^3 u_0}{\partial x\partial\phi^2}+\frac{\partial^2 u_0}{\partial x^2}\left(\frac{\partial w_0}{\partial x}-\frac{\partial^2 v_0}{\partial x\partial\phi}\right)+\left(1+\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial^2 w_0}{\partial x^2}-\frac{\partial^3 v_0}{\partial x^2\partial\phi}\right)\right)$$
$$+\frac{M_{x\phi}^o+M_{x\phi}^*}{r}\left\{\begin{array}{l}\dfrac{\partial^3 u_0}{\partial x^2\partial\phi}+\dfrac{1}{r}\left(\dfrac{\partial^3 v_0}{\partial x\partial\phi^2}-\dfrac{\partial^2 w_0}{\partial x\partial\phi}\right)\dfrac{\partial u_0}{\partial x}\\+\left(\dfrac{1}{r}\dfrac{\partial^2 u_0}{\partial x\partial\phi}+\dfrac{\partial^2 v_0}{\partial x^2}\right)\left(\dfrac{\partial^2 v_0}{\partial x\partial\phi}-\dfrac{\partial w_0}{\partial x}\right)+\left(r+\dfrac{\partial v_0}{\partial\phi}-w_0\right)\dfrac{\partial^3 v_0}{\partial x^3}\end{array}\right\}$$
$$+\left(N_x^o+N_x^*-P\right)\left(1+\frac{1}{r}\frac{\partial v_0}{\partial\phi}-\frac{w_0}{r}\right)\frac{\partial^2 w_0}{\partial x^2}+\frac{N_\phi^o+N_\phi^*}{r}\left(1+\frac{1}{r}\frac{\partial v_0}{\partial\phi}+\frac{1}{r}\frac{\partial^2 w_0}{\partial\phi^2}\right)$$
$$+\frac{N_{x\phi}^o+N_{x\phi}^*}{r}\left(2+\frac{\partial u_0}{\partial x}+\frac{1}{r}\frac{\partial v_0}{\partial\phi}-\frac{w_0}{r}\right)\left(\frac{\partial v_0}{\partial x}+\frac{\partial^2 w_0}{\partial x\partial\phi}\right)$$
$$+p\left(\frac{\partial u_0}{\partial x}\left(1+\frac{1}{r}\frac{\partial v_0}{\partial\phi}-\frac{w_0}{r}\right)+\frac{1}{r}\left(w_0+\frac{\partial^2 w_0}{\partial\phi^2}\right)\right)=0, \qquad (33)$$

$$\left(\frac{M_x^o+M_x^*}{r}\left(1+\frac{1}{r}\frac{\partial v_0}{\partial\phi}-\frac{w_0}{r}\right)+\frac{M_\phi^o+M_\phi^*}{r}\left(1+\frac{\partial u_0}{\partial x}\right)\right)\left(\frac{\partial v_0}{\partial x}+\frac{\partial^2 w_0}{\partial x\partial\phi}\right)$$
$$+\frac{M_{x\phi}^o+M_{x\phi}^*}{r}\left(\left(1+\frac{\partial u_0}{\partial x}\right)\left(\frac{1}{r}\frac{\partial v_0}{\partial x}-\frac{1}{r}\frac{\partial^2 w_0}{\partial\phi^2}\right)+\frac{\partial^2 w_0}{\partial x^2}\left(1+\frac{1}{r}\frac{\partial v_0}{\partial\phi}-\frac{w_0}{r}\right)\right)$$
$$+\left(N_{x\phi}^o+N_{x\phi}^*\right)\left(\frac{\partial u_0}{\partial x}+\frac{1}{r}\frac{\partial v_0}{\partial\phi}-\frac{w_0}{r}\right)=0. \qquad (34)$$

To further reduce these equations to a system of partial differential equations to expressions involving only the displacements of the middle surface, it is necessary to substitute the results from equations (2), (4), and (16). This results in a set of simultaneous partial differential equations for the mid-surface deflections for a laminated cylindrical shell under constant uniform lateral and axial compression. These equations are further simplified by assuming that the deformation is axisymmetric. This assumption is based on the facts that:

(1) the load and boundary conditions are independent of ϕ,

(2) the material used is uniform and homogeneous in the ϕ-direction, and

(3) the material stiffnesses, A_{ij}, B_{ij}, and D_{ij}, are independent of x, ϕ, and z.

Consequently, the circumferential displacement, v_0, is zero or constant, and that u_0 and w_0 are functions of x alone (i.e.: independent of ϕ). Thus, all partial derivatives of u_0 and w_0 with respect to ϕ vanish. Then equations (31) through (34) using equations (2), (4), and (16) reduce to a set of four simultaneous nonlinear ordinary differential equations in the mid-surface displacements, u_0 and w_0. These equations are:

$$A_{11}u_0'' - B_{11}w_0''' - \left[\frac{\left(A_{12} + N_x^* - N_\phi^* - P\right)}{r} + p\right]w_0'$$
$$+\left(\frac{N_\phi^* + A_{12} - A_{11}}{r} - p\right)u_0'w_0' - \frac{A_{11}}{r}u_0''w_0 + \frac{B_{11} - B_{12}}{r}w_0'w_0'' + \frac{B_{11}}{r}w_0 w_0''' + \frac{2A_{12} - A_{22}}{r^2}w_0 w_0'$$
$$+\frac{A_{12}}{r}u_0'^2 w_0' - \frac{A_{22}}{r^2}u_0' w_0 w_0' - \frac{B_{12}}{r}u_0' w_0' w_0'' = 0\,, \tag{35}$$

$$\left(A_{16} + \frac{B_{16}}{r}\right)u_0'' - \frac{1}{r}\left[A_{26} + \frac{B_{26}}{r} - 2N_{x\phi}^*\right]w_0' - \left(B_{16} + \frac{D_{16}}{r}\right)w_0'''$$
$$+\frac{1}{r}\left(\frac{M_{x\phi}^*}{r} - N_{x\phi}^* - 2A_{16}\right)u_0'w_0' - \frac{1}{r}\left(A_{16} + \frac{B_{16}}{r}\right)u_0''w_0$$
$$-\frac{1}{r^2}\left(3A_{26} + \frac{B_{26}}{r}\right)w_0 w_0' + \frac{1}{r}\left(B_{16} + \frac{D_{16}}{r}\right)w_0 w_0''' + \frac{2B_{16}}{r}w_0'w_0''$$
$$-\frac{1}{r}\left(A_{16} - \frac{B_{16}}{r}\right)u_0'^2 w_0' + \frac{1}{r^2}\left(A_{26} - \frac{B_{26}}{r}\right)u_0' w_0 w_0' + \frac{1}{r}\left(B_{16} - \frac{D_{16}}{r}\right)u_0' w_0' w_0'' = 0, \tag{36}$$

$$
B_{11}u_0''' + \left(\frac{A_{12}}{r} + p\right)u' - D_{11}w_0^{iv} - \left(\frac{2B_{12} + M_x^* - M_\phi^*}{r} + N_x^* - P\right)w_0'' - \frac{1}{r}\left(\frac{A_{12}}{r} - p\right)w_0
$$

$$
-\frac{B_{11}}{r}u_0'''w_0 + \frac{B_{12} + M_\phi^* - 2B_{11}}{r}u_0''w_0' + \left(\frac{B_{12} - B_{11} + M_\phi^*}{r} + A_{11}\right)u_0'w_0'' - \frac{p}{r}u_0'w_0
$$

$$
+\frac{D_{11}}{r}w_0w_0^{iv} + \frac{2D_{11}}{r}w_0'w_0''' + \frac{1}{r}\left(\frac{2B_{12} - B_{26}}{r} - A_{12} - N_x^* + P\right)w_0w_0''
$$

$$
+\left(\frac{D_{11} - D_{12}}{r} - B_{11}\right)w_0''^{\,2} + \frac{2B_{12} - B_{26}}{r^2}w_0'^{\,2} + \frac{2B_{12}}{r}u_0'u_0''w_0' - \frac{D_{12}}{r}u_0'w_0'w_0'''
$$

$$
-\frac{1}{r}\left(A_{11} + \frac{B_{26}}{r}\right)u_0'w_0w_0'' - \frac{D_{12}}{r}u_0'w_0''^{\,2} - \frac{B_{26}}{r^2}u_0'w_0'^{\,2} + \frac{B_{12}}{r}u_0'^{\,2}w_0'' - \frac{B_{26}}{r^2}u_0''w_0w_0'
$$

$$
-\frac{D_{12}}{r}u_0''w_0'w_0'' + \frac{A_{12}}{r^2}w_0^{\,2}w_0'' + \frac{B_{11}}{r}w_0w_0''^{\,2} + \frac{N_\phi^*}{r} = 0\,, \tag{37}
$$

$$
\left(\frac{B_{16} + M_{x\phi}^*}{r} + N_{x\phi}^*\right)u_0' + \left(M_{x\phi}^* - \frac{D_{16}}{r}\right)w_0'' + \frac{1}{r}\left(N_{x\phi}^* - \frac{B_{26}}{r}\right)w_0 + \frac{M_{x\phi}^*}{r}
$$

$$
+\frac{1}{r}\left(A_{16} - A_{26} - \frac{B_{26}}{r^2}\right)u_0'w_0 - \frac{D_{16}}{r}u_0'w_0'' - \frac{B_{16} + B_{26} - M_{x\phi}^*}{r}w_0w_0'' + \left(A_{16} + \frac{B_{16}}{r}\right)u_0'^{\,2}
$$

$$
-\frac{A_{26}}{r^2}w_0^{\,2} - D_{16}w_0''^{\,2} + \frac{B_{26}}{r^2}w_0^{\,2}w_0'' + \frac{D_{16}}{r}w_0w_0''^{\,2} - \frac{B_{16}}{r}u_0'w_0w_0'' = 0, \tag{38}
$$

where $(\ )'$ indicates ordinary derivative with respect to the *x*-coordinate. As discussed, equations (35) through (38) form a system of four simultaneous nonlinear ordinary differential equations in two unknowns. This system can be further reduced, to a system of two simultaneous ordinary linear differential equations, through the following process: (*i*) eliminate the w_0w_0' term from equation (36) by substituting from equation (35); (*ii*) differentiate the new equation with respect to *x*; (*iii*) eliminate the $w_0'w_0'''$ term from equation (37) by substituting from the new equation; and linearize the system by application of the following assumptions: (a) the in-plane normal strains, ε_x^0 and ε_ϕ^0, are very small compared to unity, and (b) the out of plane rotations, w_0', are small. The final system is:

$$\left(\frac{B_{16}+M^*_{x\phi}}{r}+N^*_{x\phi}\right)u_0' + \left(M^*_{x\phi}-\frac{D_{16}}{r}\right)w_0'' + \frac{1}{r}\left(N^*_{x\phi}-\frac{B_{26}}{r}\right)w_0 + \frac{M^*_{x\phi}}{r} = 0, \tag{39}$$

$$\left(B_{11}+\frac{2D_{11}\xi_1}{(\xi_4+\xi_5)}\right)u_0''' + \left(\frac{A_{12}}{r}+p\right)u_0'$$
$$+\left(\frac{2\xi_2}{(\xi_4+\xi_5)}-1\right)D_{11}w_0^{iv} + \left(\frac{2D_{11}\xi_3}{(\xi_4+\xi_5)} - \frac{2B_{12}+M^*_x-M^*_\phi}{r} - N^*_x + P\right)w_0''$$
$$+\frac{1}{r}\left(\frac{A_{12}}{r}-p\right)w_0 + \frac{N^*_\phi}{r} = 0, \tag{40}$$

where ξ_i are constants that involve the material properties, shell radius, prestresses, and the shell loads. They are:

$$\xi_1 = \left(A_{16}+\frac{B_{16}}{r}-\left(\frac{A_{26}r+B_{26}}{A_{22}-2A_{12}}\right)A_{11}\right)$$
$$\xi_2 = \left(\left(\frac{A_{26}r+B_{26}}{A_{22}-2A_{12}}\right)B_{11}-B_{16}-\frac{D_{16}}{r}\right)$$
$$\xi_3 = \frac{1}{r}\left(\left(\frac{A_{26}r+B_{26}}{A_{22}-2A_{12}}\right)\left[A_{12}+N^*_x-N^*_\phi+pr-P\right]-\left[A_{26}-2N^*_{x\phi}\right]-\frac{B_{26}}{r}\right)$$
$$\xi_4 = \frac{1}{r}\left(2B_{16}-\left(\frac{A_{26}r+B_{26}}{A_{22}-2A_{12}}\right)(B_{11}-B_{12})\right)$$
$$\xi_5 = \frac{-\xi_2}{r}. \tag{41}$$

Thus, the deflection of the shell is determined by solving equations (39) and (40).

Solution of the Governing Equations

The solution of the governing equations of a laminated cylindrical shell under constant uniform lateral and axial compression is obtained as discussed in the following. Equation (39) can be rearranged such that all terms involving the radial displacement are grouped on one side of the equality while the axial displacement terms are on the other side. Writing this gives:

$$\left(\frac{B_{16}+M^*_{x\phi}}{r}+N^*_{x\phi}\right)u_0' = \left(\frac{D_{16}}{r}-M^*_{x\phi}\right)w_0'' + \frac{1}{r}\left(\frac{B_{26}}{r}-N^*_{x\phi}\right)w_0 - \frac{M^*_{x\phi}}{r}. \tag{42}$$

Next, define the constants α, β, and γ as:

$$\alpha = \frac{D_{16}}{B_{16} + M_{x\phi}^* + N_{x\phi}^* r} - r\gamma \,,$$

$$\beta = \frac{B_{26} - N_{x\phi}^* r}{r\left(B_{16} + M_{x\phi}^* + N_{x\phi}^* r\right)} \,,$$

$$\gamma = \frac{M_{x\phi}^*}{B_{16} + M_{x\phi}^* + N_{x\phi}^* r} . \qquad (43)$$

Then, equation (42), its derivative and anti-derivative become:

$$u_0' = \alpha w_0'' + \beta w_0 - \gamma \,,$$

$$u_0''' = \alpha w_0^{iv} + \beta w_0'' \,,$$

$$u_0 = \alpha w_0' + \beta \int_x w_0(\chi) d\chi - \gamma x + C_5 \,, \qquad (44)$$

where C_5 is the constant of integration. Substituting the results given in equation (44) into equation (40) yields a single fourth order ordinary differential equation to be solved for the radial displacement, w_0. After collecting terms, this equation is:

$$\left(\left(B_{11} + \frac{2D_{11}\xi_1}{(\xi_4 + \xi_5)r}\right)\alpha + \frac{2\xi_2 D_{11}}{(\xi_4 + \xi_5)r} - D_{11}\right) w_0^{iv}$$

$$+\left(\left(B_{11} + \frac{2D_{11}\xi_1}{(\xi_4 + \xi_5)r}\right)\beta + \left(\frac{A_{12}}{r} - p\right)\alpha + \frac{2D_{11}\xi_3}{(\xi_4 + \xi_5)r} - \frac{2B_{12} + M_x^* - M_\phi^*}{r} - N_x^* + P\right) w_0''$$

$$+\frac{1}{r}\left(\frac{A_{12}}{r} - p\right)(1 + \beta r) w_0 + \frac{N_\phi^* - (A_{12} - pr)\gamma}{r} = 0. \qquad (45)$$

To simplify the notation in the solution of the governing differential equation for the radial displacement, introduce the following constants:

$$a_0 = \frac{1}{\left(B_{11} + \frac{2D_{11}\xi_1}{(\xi_4 + \xi_5)r}\right)\alpha + \frac{2\xi_2 D_{11}}{(\xi_4 + \xi_5)r} - D_{11}} \,,$$

$$a_1 = \left(\left(B_{11} + \frac{2D_{11}\xi_1}{(\xi_4 + \xi_5)r}\right)\beta + \left(\frac{A_{12}}{r} - p\right)\alpha + \frac{2D_{11}\xi_3}{(\xi_4 + \xi_5)r} - \frac{2B_{12} + M_x^* - M_\phi^*}{r} - N_x^* + P\right) a_0 \,,$$

$$a_2 = \frac{1}{r}\left(\frac{A_{12}}{r} - p\right)(1 + \beta r) a_0 \,,$$

$$a_3 = \frac{N_\phi^* - (A_{12} - pr)\gamma}{r} a_0 . \qquad (46)$$

Then, equation (45) can be rewritten as:

$$\frac{d^4 w_0}{dx^4} + a_1 \frac{d^2 w_0}{dx^2} + a_2 w_0 + a_3 = 0. \tag{47}$$

This, then, is the equation to be solved for the radial displacement of a cylindrical laminated shell under constant uniform lateral and axial compression. This is a non-homogeneous linear ordinary differential equation with constant coefficients. At this point in the solution no comments can be made on the relative magnitude of the coefficients, a_i. Therefore, the solution is:

$$\begin{aligned} w_0(x) &= C_1 \exp\left(\sqrt{\frac{b-a_1}{2}}x\right) + C_2 \exp\left(-\sqrt{\frac{b-a_1}{2}}x\right) \\ &\quad + C_3 \exp\left(\sqrt{\frac{-b-a_1}{2}}x\right) + C_4 \exp\left(-\sqrt{\frac{-b-a_1}{2}}x\right) - \frac{a_3}{a_2}, \end{aligned} \tag{48}$$

where a_i are defined in equation (46), $b = \sqrt{a_1^2 - 4a_2}$, and C_i are the constants to be determined from the boundary conditions. With $w_0(x)$ known from equation (48) the axial displacement can be written as:

$$\begin{aligned} u_0(x) &= \left\{ \alpha\sqrt{\frac{b-a_1}{2}} + \beta\sqrt{\frac{2}{b-a_1}} \right\} \left\{ C_1 \exp\left(\sqrt{\frac{b-a_1}{2}}x\right) - C_2 \exp\left(-\sqrt{\frac{b-a_1}{2}}x\right) \right\} \\ &\quad + \left\{ \alpha\sqrt{\frac{-b-a_1}{2}} + \beta\sqrt{\frac{2}{-b-a_1}} \right\} \left\{ C_3 \exp\left(\sqrt{\frac{-b-a_1}{2}}x\right) - C_4 \exp\left(-\sqrt{\frac{-b-a_1}{2}}x\right) \right\} \\ &\qquad - \left(\gamma + \beta\frac{a_3}{a_2}\right)x + C_5. \end{aligned} \tag{49}$$

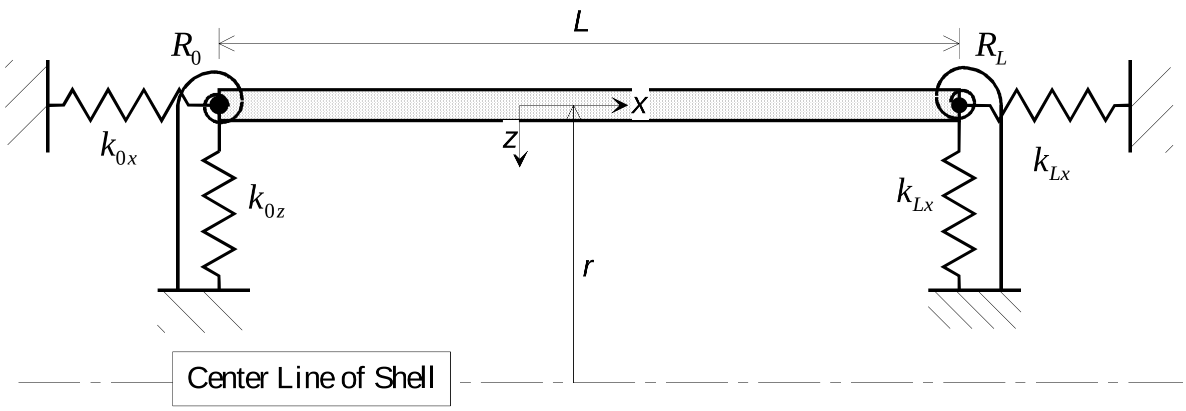

Figure 4. Shell Boundary Conditions

The boundary conditions for the shell may be defined in general by a system of springs that define the stiffness of the boundary. Figure 4 contains a schematic drawing of the proposed elastic foundation for the shell. Note that the foundation shown includes linear and torsional springs. The constants k_{0z}, k_{0x} k_{Lz}, and k_{Lx} represent the linear stiffness of the boundaries in the *z*- and *x*-directions, respectively, at x=0 and x=L, respectively. Further, the constants R_0 and R_L represent the rotational or torsional stiffness of the boundaries at x=0 and x=L, respectively. The boundary conditions are:

$$(1)\; k_{0z} w_0(0) = \bar{N}_{xz}\big|_{x=0} - \frac{1}{r}\frac{\partial \bar{M}_{x\phi}}{\partial \phi} \Rightarrow \bar{N}_{xz}\big|_{x=0} \Rightarrow \frac{\mathrm{d}\bar{M}_x}{\mathrm{d}x}\bigg|_{x=0}$$

$$\Rightarrow \frac{\mathrm{d}M_x^o}{\mathrm{d}x}\bigg|_{x=0} - \left(1 - \frac{w_0(0)}{r}\right)\frac{M_x^o + M_x^*}{r}\bigg|_{x=0} w_0'(0) \Rightarrow \frac{\mathrm{d}M_x^o}{\mathrm{d}x}\bigg|_{x=0} - \frac{M_x^*}{r}\bigg|_{x=0} w_0'(0)$$

$$(2)\; R_0 w_0'(0) = \bar{M}_x\big|_{x=0} \Rightarrow \left(M_x^o + M_x^*\right)\big|_{x=0}\left(1 - \frac{w_0(0)}{r}\right) \Rightarrow M_x^o\big|_{x=0} + M_x^*\left(1 - \frac{w_0(0)}{r}\right)$$

$$(3)\; k_{Lz} w_0(L) = \bar{N}_{xz}\big|_{x=L} \Rightarrow \frac{\mathrm{d}M_x}{\mathrm{d}x}\bigg|_{x=L} - \frac{M_x^*}{r}\bigg|_{x=L} w_0'(L)$$

$$(4)\; R_L w_0'(0) = \bar{M}_x\big|_{x=L} \Rightarrow \left(M_x^o + M_x^*\right)\big|_{x=L}\left(1 - \frac{w_0(L)}{r}\right) \Rightarrow M_x^o\big|_{x=L} + M_x^*\left(1 - \frac{w_0(L)}{r}\right)$$

$$(5)\; k_{0x} u_0(0) = \bar{N}_x\big|_{x=0} \Rightarrow \left(N_x^o\big|_{x=0} + N_x^* - P\right)\left(1 - \frac{w_0(0)}{r}\right) \Rightarrow N_x^o\big|_{x=0} + \left(N_x^* - P\right)\left(1 - \frac{w_0(0)}{r}\right)$$

$$(6)\; k_{Lx} u_0(0) = \bar{N}_x\big|_{x=L} \Rightarrow N_x^o\big|_{x=L} + \left(N_x^* - P\right)\left(1 - \frac{w_0(L)}{r}\right)$$

where

$$M_x^o(x) = \left(B_{11}\alpha - D_{11}\right) w_0''(x) + \left(B_{11}\beta - \frac{B_{12}}{r}\right) w_0(x) - B_{11}\gamma,$$

$$N_x^o(x) = \left(A_{11}\alpha - B_{11}\right) w_0''(x) + \left(A_{11}\beta - \frac{A_{12}}{r}\right) w_0(x) - A_{11}\gamma.$$

These equations may now be expressed in matrix form such that:

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{Bmatrix} \qquad (50)$$

where k_{ij} and f_i are constants, which are complex expressions involving the material constants, radius of the shell, applied loads, and preloads, and C_i are the constants to be determined. The expressions for k_{ij} and f_i are given in the appendix.

Special cases of the Solution Obtained

An important aspect of the previous analysis is that the governing equation for the displacement of the middle surface of a composite cylindrical shell subject to constant lateral and axial compression and, hence, the solution is valid for two special load cases: (*i*) lateral pressure only, and (*ii*) axial compression only.

i. Laminated Cylindrical shell under Uniform Lateral Pressure:

In this case the only external load acting on the shell is the uniform lateral compression, $p_z = p$. Therefore, the expression for ξ_3 in equation (41) becomes:

$$\xi_3 = \frac{1}{r}\left[\left(\frac{A_{26}r + B_{26}}{A_{22} - 2A_{12}}\right)\left(A_{12} + N_x^* - N_\phi^* + pr\right) - \left(A_{26} - 2N_{x\phi}^*\right) - \frac{B_{26}}{r}\right] \qquad (51)$$

Further, a_1 in equation (46) becomes:

$$a_1 = \left(\left(B_{11} + \frac{2D_{11}\xi_1}{(\xi_4 + \xi_5)r}\right)\beta + \left(\frac{A_{12}}{r} - p\right)\alpha + \frac{2D_{11}\xi_3}{(\xi_4 + \xi_5)r} - \frac{2B_{12} + M_x^* - M_\phi^*}{r} - N_x^*\right)a_0. \qquad (52)$$

The solution is then obtained as in the general caseand agrees with reference [6].

ii. Laminated Cylindrical shell under Uniform Axial Compression:

In this case the only external load acting on the shell is the uniform axial compression, $N_x = P$. In a manner similar to that in the previous case, the expression for ξ_3 in equation (41) and a_i in equation (46) simplify to become:

$$\xi_3 = \frac{1}{r}\left[\left(\frac{A_{26}r + B_{26}}{A_{22} - 2A_{12}}\right)\left(A_{12} + N_x^* - N_\phi^* - P\right) - \left(A_{26} - 2N_{x\phi}^*\right) - \frac{B_{26}}{r}\right] \qquad (53)$$

$$a_1 = \left[\left(B_{11} + \frac{2D_{11}\xi_1}{(\xi_4 + \xi_5)}\right)\beta + \frac{A_{12}}{r}\alpha + \frac{2D_{11}\xi_3}{(\xi_4 + \xi_5)} - \frac{2B_{12} + M_x^* - M_\phi^*}{r} - N_x^* + P\right]a_0,$$

$$a_2 = \frac{A_{12}}{r^2}\left(1 + \beta r\right)a_0,$$

$$a_3 = \frac{N_\phi^* - A_{12}\gamma}{r}a_0. \qquad (54)$$

and, again, the solution is as obtained in the general case.

Special Construction Techniques:

If the laminated cylindrical shell is constructed such that the laminate is (1) symmetric, then the coupling stiffnesses are zero, i.e.: $B_{ij} = 0$, i, j = 1, 2, 6; and (2) balanced, then the 1-6 and 2-6 axial stiffnesses are zero, i.e.: $A_{16} = A_{26} = 0$. For this specific design the constants ξ_i in equation (41), α, β, and γ in equation (43), and a_i in equation (46) are greatly simplified. For this case they become:

$$\left.\begin{aligned}
&\xi_1 = 0\,,\\
&\xi_2 = -\frac{D_{16}}{r}\,,\\
&\xi_3 = \frac{2N^*_{x\phi}}{r}\,,\\
&\xi_4 = 0\,,\\
&\xi_5 = \frac{-\xi_2}{r}\,,\\
&\alpha = \frac{D_{16}}{M^*_{x\phi} + N^*_{x\phi} r} - r\gamma\,,\\
&\beta = \frac{-N^*_{x\phi}}{M^*_{x\phi} + N^*_{x\phi} r}\,,\\
&\gamma = \frac{M^*_{x\phi}}{M^*_{x\phi} + N^*_{x\phi} r}\,,
\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}
&a_1 = \frac{1}{3}\left(\frac{M^*_x - M^*_\phi + \left(N^*_x - P\right)r}{D_{11} r}\right.\\
&\qquad\left. - \frac{\left(A_{12} - pr\right)\left(D_{16} - M^*_{x\phi} r\right)}{D_{11} r\left(M^*_{x\phi} + N^*_{x\phi} r\right)} - \frac{4N^*_{x\phi}}{D_{16}}\right),\\
&a_2 = \frac{M^*_{x\phi}\left(pr - A_{12}\right)}{3D_{11}\left(M^*_{x\phi} + N^*_{x\phi} r\right)}\,,\\
&a_3 = \left(\frac{1}{3D_{11} r}\right)\left(\frac{M^*_{x\phi}\left(A_{12} - pr\right)}{\left(M^*_{x\phi} + N^*_{x\phi} r\right)} - N^*_\phi\right).
\end{aligned}\right. \tag{55}$$

The solution then follows the previous pattern.

NUMERICAL EXAMPLE

The solution obtained was evaluated for various materials, stacking sequences, pre-stresses, and boundary and loading conditions. A sample of the problems addressed are shown in this section. Material property values for the solutions presented here are listed in Table 1.

Table 1. Material Property Values Used in Examples.

Material	E_1 Mpsi	E_2 Mpsi	ν_{12}	G_{12} Mpsi
E-glass/Epoxy	5.6	1.2	0.26	0.6
IM6/Epoxy	29.46	1.63	0.32	1.22
Kevlar-49/Epoxy	11.03	0.8	0.34	0.33

Effect of Pre-stress

The effect of pre-stress is shown in the analysis of a symmetric laminated composite shell of length 30 inches and diameter 10 inches. The shell in this example has twelve ply of IM6/Epoxy with a stacking sequence of $\left[2[\pm 45]/2[90]\right]_{2S}$. The shell is simply supported with lateral pressure of 3000 psi and zero axial load. A plot of the deformed midplane location for two different pre-stress values is shown in Figure 5. Note that in this figure the maximum radial displacement for a pre-stress of 1.78 lb./tow is of the order of magnitude of 10^{-6}, while the maximum radial displacement for the pre-stress of 0.178 lb./tow is of the order of magnitude of 10^{-7}. In neither case was the internal pressure sufficient to overcome the drawing in of the shell due to the pre-stress.

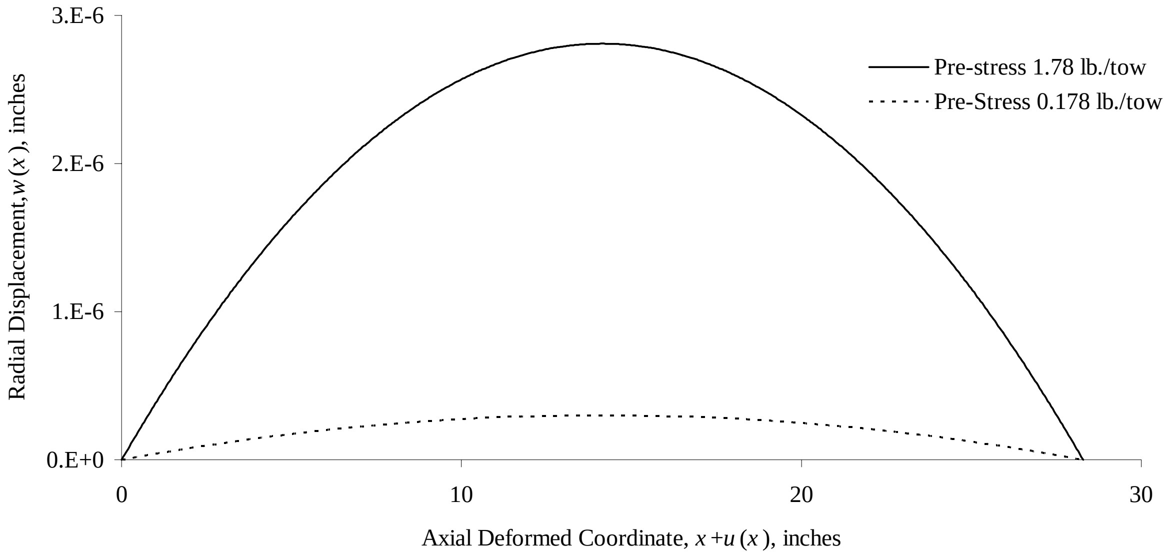

Figure 5. Plot of Deformed Midplane Location Showing Effect of Pre-Stress with Constant Loading Conditions.

Effect of Lateral Pressure and Axial Compression with Constant Pre-stress

The solution obtained was evaluated for a simply-supported 33 ply hybrid composite shells of length 10 and 5 inches and diameter 10 inches. The stacking sequence used for the hybrid laminate is asymmetric and is $\left[2[\pm 45]_{\text{E-glass/ Epoxy}} \middle/ \left[4[90/\pm 45/0/\pm 45]/90\right]_{\text{IM6/ Epoxy}} \middle/ 2[\pm 45]_{\text{Kevlar-49/ Epoxy}}\right]_T$. The position of the deformed shell midplane is shown in Figures 6 and 7. Figure 6 shows the effect of lateral pressure and axial thrust for constant pre-stress. The pre-stress for all examples in Figures 6 and 7 is 0.375 lbs./tow; the width per tow is 1 inch and the tow diameter is 0.005 inches. The effect of the length on the deformed geometry is shown in Figure 7 for a lateral pressure of 3000 psi and 100 lb. axial load.

The results shown in Figure 6 are consistent with those shown by Chaudhuri, Balaraman, and Kunukkasseril[6] for an anisotropic cylindrical shell under internal pressure. The results are also consistent with in the different loading conditions. Note that the deflection increases (for the coordinate system used, negative deflection is away from the centerline of the shell) as lateral pressure increases. The plot shows that the deformed length decreases as the axial load and lateral pressure increase, as expected with simple supports.

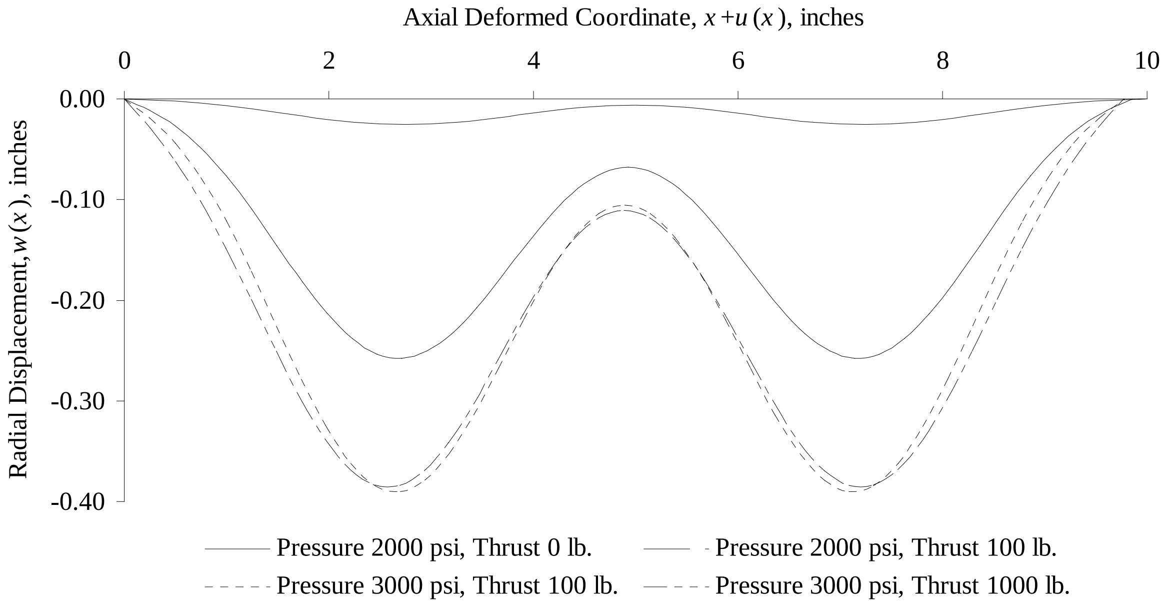

Figure 6. Deformed Midplane of Hybrid Laminated Cylindrical Shell for Various Loading Conditions with Simple Supports and a Constant Pre-Stress of 0.375 lbs./tow.

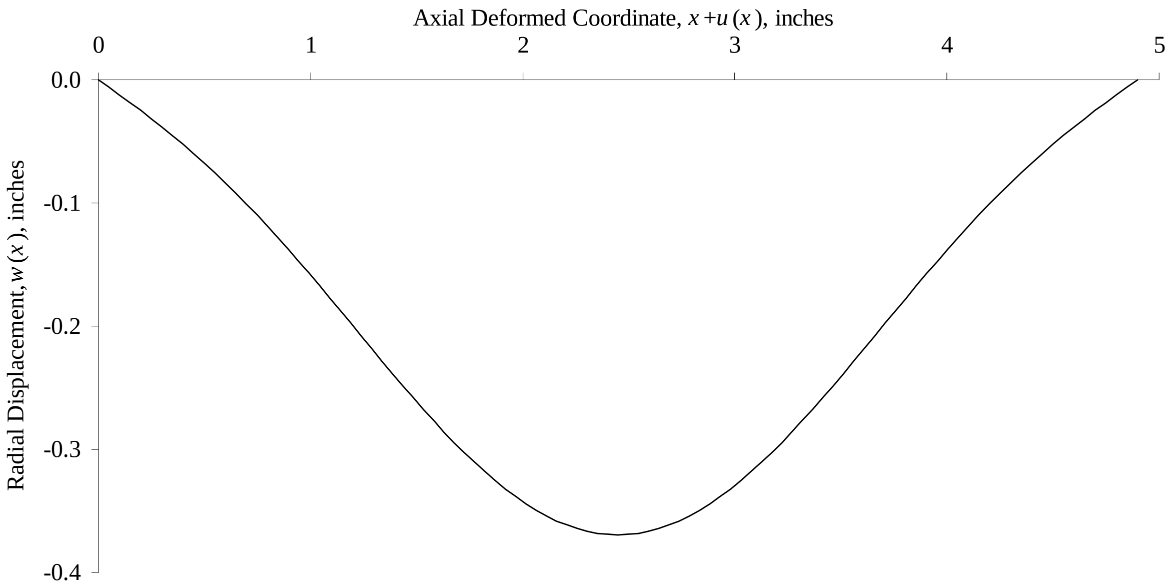

Figure 7. Deformed Midplane of Hybrid Laminated Cylindrical Shell for Length of 5 Inches, Simply Supported Boundary Conditions, Lateral Pressure of 3000 psi, and zero Axial Load.

CONCLUSIONS

A closed form solution for cylindrical laminated shells has been developed. Features of the solution address nonlinear effects due to large deformations, material orthotropy, and effects of pre-stress due to a winding process. Equations are presented to evaluate the solution with flexible or elastic boundary conditions. The solution allows analysis of shells with general stacking sequences.

Results from the numerical examples presented indicate that: an order of magnitude increase in the winding pre-stress can result in an order of magnitude increase in the shell deformation; and the deformed shape of the shell changes dramatically as the length to diameter ratio changes. Results are presented for symmetric and asymmetric laminates.

This solution will be a good tool for the SRM design and optimization of its load carrying capacity and structure weight.

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APPENDIX

The coefficients of the matrix equation (50) k_{ij}, f_j will take the form :

$$k_{11} = k_{0z} - \Omega_1 + \frac{M_X^*}{r}\sqrt{\frac{b-a_1}{2}} \quad ,$$

$$k_{12} = k_{0z} + \Omega_1 - \frac{M_x^*}{r}\sqrt{\frac{b-a_1}{2}},$$

$$k_{13} = k_{0z} - \Omega_2 + \frac{M_x^*}{r}\sqrt{\frac{-(b+a_1)}{2}},$$

$$k_{14} = k_{0z} + \Omega_2 - \frac{M_x^*}{r}\sqrt{\frac{-(b+a_1)}{2}},$$

$$k_{21} = R_0\sqrt{\frac{b-a_1}{2}} - \Omega_3 + \frac{M_x^*}{r},$$

$$k_{22} = -R_0\sqrt{\frac{b-a_1}{2}} - \Omega_3 + \frac{M_x^*}{r},$$

$$k_{23} = R_0\sqrt{\frac{-(b+a_1)}{2}} + \Omega_4 + \frac{M_x^*}{r},$$

$$k_{24} = -R_0\sqrt{\frac{-(b+a_1)}{2}} + \Omega_4 + \frac{M_x^*}{r},$$

$$k_{31} = \left(k_{Lz} - \Omega_1 + \frac{M_x^*}{r}\sqrt{\frac{b-a_1}{2}}\right)\exp\left(\sqrt{\frac{b-a_1}{2}}L\right),$$

$$k_{32} = \left(k_{Lz} + \Omega_1 - \frac{M_x^*}{r}\sqrt{\frac{b-a_1}{2}}\right)\exp\left(-\sqrt{\frac{b-a_1}{2}}L\right),$$

$$k_{33} = \left(k_{Lz} - \Omega_2 + \frac{M_x^*}{r}\sqrt{\frac{-(b+a_1)}{2}}\right)\exp\left(\sqrt{\frac{-(b+a_1)}{2}}L\right),$$

$$k_{34} = \left(k_{Lz} + \Omega_2 - \frac{M_x^*}{r}\sqrt{\frac{-(b+a_1)}{2}}\right)\exp\left(-\sqrt{\frac{-(b+a_1)}{2}}L\right),$$

$$k_{41} = \left(R_L\sqrt{\frac{b-a_1}{2}} - \Omega_3 + \frac{M_x^*}{r}\right)\exp\left(\sqrt{\frac{b-a_1}{2}}L\right),$$

$$k_{42} = \left(-R_L \sqrt{\frac{b-a_1}{2}} - \Omega_3 + \frac{M_x^*}{r} \right) \exp\left(-\sqrt{\frac{b-a_1}{2}} L \right) ,$$

$$k_{43} = \left(R_L \sqrt{\frac{-(b+a_1)}{2}} + \Omega_4 + \frac{M_x^*}{r} \right) \exp\left(\sqrt{\frac{-(b+a_1)}{2}} L \right),$$

$$k_{44} = \left(-R_L \sqrt{\frac{-(b+a_1)}{2}} + \Omega_4 + \frac{M_x^*}{r} \right) \exp\left(-\sqrt{\frac{-(b+a_1)}{2}} L \right),$$

$$k_{15} = k_{25} = k_{35} = k_{45} = 0,$$

$$k_{51} = k_{0x} \left(\alpha \sqrt{\frac{b-a_1}{2}} + \beta \sqrt{\frac{2}{b-a_1}} \right) + \Omega_5,$$

$$k_{52} = -k_{0x} \left(\alpha \sqrt{\frac{b-a_1}{2}} + \beta \sqrt{\frac{2}{b-a_1}} \right) + \Omega_5,$$

$$k_{53} = k_{0x} \left(\alpha \sqrt{\frac{-(b+a_1)}{2}} + \beta \sqrt{\frac{-2}{b+a_1}} \right) + \Omega_6,$$

$$k_{54} = -k_{0x} \left(\alpha \sqrt{\frac{-(b+a_1)}{2}} + \beta \sqrt{\frac{-2}{b+a_1}} \right) + \Omega_6,$$

$$k_{55} = k_{0x} \quad ,$$

and

$$f_1 = k_{0z} \left(\frac{a_3}{a_2} \right) \quad ,$$

$$f_2 = \left(\frac{M_x^* - B_{11} \beta r + B_{12}}{r} \right) \left(\frac{a_3}{a_2} \right) + M_x^* - B_{11} \gamma ,$$

$$f_3 = k_{Lz} \left(\frac{a_3}{a_2} \right),$$

$$f_4 = \left(\frac{M_x^* - B_{11} \beta r + B_{12}}{r} \right) \left(\frac{a_3}{a_2} \right) + M_x^* - B_{11} \gamma ,$$

$$f_5 = \left(\frac{M_x^* + A_{12} - P - A_{11}\beta r}{r} \right) \left(\frac{a_3}{a_2} \right) + N_x^* - P - A_{11}\gamma \ ,$$

where,

$$\Omega_1 = \left(B_{11}\alpha - D_{11} \right) \left(\frac{b - a_1}{2} \right)^{\frac{3}{2}} + \left(\frac{B_{11}\beta r - B_{12}}{r} \right) \left(\frac{b - a_1}{2} \right)^{\frac{1}{2}} ,$$

$$\Omega_2 = \left(B_{11}\alpha - D_{11} \right) \left(\frac{-b - a_1}{2} \right)^{\frac{3}{2}} + \left(\frac{B_{11}\beta r - B_{12}}{r} \right) \left(\frac{-b - a_1}{2} \right)^{\frac{1}{2}} ,$$

$$\Omega_3 = \left(B_{11}\alpha - D_{11} \right) \left(\frac{b - a_1}{2} \right) + \left(\frac{B_{11}\beta r - B_{12}}{r} \right) \ ,$$

$$\Omega_4 = \left(B_{11}\alpha - D_{11} \right) \left(\frac{b + a_1}{2} \right) - \left(\frac{B_{11}\beta r - B_{12}}{r} \right) \ ,$$

$$\Omega_5 = N_x^* - P - \left(A_{11}\alpha - B_{11} \right) \left(\frac{b - a_1}{2} \right) - \left(\frac{A_{11}\beta r - A_{12}}{r} \right) \ ,$$

$$\Omega_6 = N_x^* - P + \left(A_{11}\alpha - B_{11} \right) \left(\frac{b + a_1}{2} \right) - \left(\frac{A_{11}\beta r - A_{12}}{r} \right) \ .$$