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Symbolic Calculation for Free Convection In a Circular Cavity with Constant Heat Flux

Kamyar Mansour

(Department of Aerospace Engineering and New Technologies Research Center, Amirkabir University of Technology, Tehran , Iran, 15875-4413
and
Flow Research and Engineering, 22166 Rae Lane, Cupertino, CA 95014)
mansour@aut.ac.ir

ABSTRACT
We consider the two-dimensional problem of steady natural convection in a circular cavity with constant volumetric heat flux filled with viscous fluid subject to cosine temperature variation on the boundary. The solution is expanded for low rayleigh number and extended to 16 terms by computer. Analysis of these expansions allows the exact computation for arbitrarily accuracy up to 50000 figures. Although the range of the radius of convergence is small but pade approximation leads our result to be good even for higher value of the similarity parameter

KEYWORDS:
Natural convection, symbolic calculation, nonlinear equation

INTRODUCTION
We consider the two-dimensional problem of steady natural convection in a Horizontal Cylinder with constant volumetric heat flux filled with viscous fluid subject to cosine temperature variation on the boundary this temperature variation corresponds to a linear gradient in a highly conductive solid material containing the cavity (By assuming finite Prandtl number to be order of unity). The solution is expanded in powers of Rayleigh number and the series extended by means of symbolic calculation up to 16 terms. Analysis of these expansions allows the exact computation for arbitrarily accuracy up to 50000 figures. Although the range of exactness is almost the same order as of the radius of convergence but Pade approximation lead our result to be good even for much higher value of the similarity parameter. Previous works on the same problem but without constant volumetric heat flux can be divided into four categories: First experiments conducted by [1], [2]. Second, theoretical analysis performed by [3], [4], [5], [6]. They expanded a perturbation about pure conduction in a cavity. Third, boundary layer analysis done by references [7], [8] Fourth, numerical solutions calculated by [9], [10]. In the past, the range of applicability of Stokes expansion was restricted to the small values of Rayleigh number. However, this work enables us to go beyond that range all the way to finite value.

STATEMENT OF PROBLEM
We consider the two-dimensional problem of steady natural convection cavity with having uniform heat generation q''' .as sketched in figure 1.

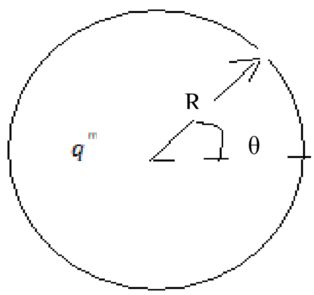

Fig.1 Geometry of the Problem

. We introduced cylindrical coordinates (r,θ) θ measured from horizontal axis as is shown in Figure1. We mainly follow Mansour's notation [8]. Then the velocity component U, V, W in the fluid are functions of r and θ only. The continuity equation can be satisfied by introducing a Stokes stream function ψ for the cross flow the Prandtl number Pr, Raleigh number Ra ,Grashof number G are defined as:

$$Ra = \frac{\beta g \Delta T_0 R^3}{\kappa \nu} = G\,\mathrm{Pr}\ ,\ Pr = \frac{\upsilon}{\kappa}\ G = \frac{\beta g \Delta T_0 R^3}{\nu^2}$$

Here β is the coefficient of thermal expansion and g the acceleration due to gravity ν is the kinematics viscosity and κ is thermal conductivity and ΔT_0 is calculated from uniform heat generation q''', $\Delta T_0 \sim \frac{q''' R^2}{\alpha}$ is somehow certain gradient of temperature across the cavity. Then

$$\nabla^4\psi = GPr(\frac{\partial T}{\partial r}\cos\theta - \frac{\partial T}{\partial \theta}\frac{1}{r}\sin\theta) + \frac{1}{Pr}\left[\frac{1}{r}\left(-\frac{\partial\psi}{\partial\theta} + \frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}\right)\nabla^2\psi\right] \quad (1)$$

$$\frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial T}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial T}{\partial\theta}\right) + 1 = \nabla^2 T \quad (2)$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$

The corresponding boundary conditions expressing the impermeability of the wall, the no-slip conditions, and imposed temperature distribution are respectively:

$$\psi = 0, \qquad at\, r = 1$$

$$\partial\psi/\partial r = 0 \qquad at\, r = 1$$

$$T = \frac{1}{4} + cos(\theta) \qquad at\, r = 1,$$

SERIES DERIVATION AND COMPUTER EXTENSION

In this section we consider the case of a fixed Prandtl number equal to unity (because of weak dependence on Pr). We return to eqs. Defined by (1) and (2) It is possible to solve these equations as a regular expansion in the Grashof number (which for Pr = 1 is the same as the Rayleigh number) λ, i.e.

$$T = T_0 + T_1\lambda + T_2\lambda^2 + \cdots = \sum_{n=0}^{\infty} T_n\lambda^n. \quad (3)$$

$$\psi = \psi_1\lambda + \psi_2\lambda^2 + \cdots = \sum_{n=1}^{\infty} T_n\lambda^n. \quad (4)$$

It is easy to show that $\psi_0 = 0$. Therefore, the basic solution is the state of simple conduction. Substituting and equating like powers of λ yields this sequence of successive linear equations, together with boundary conditions we obtain:

$$T_0 = (\tfrac{1}{4})r(r + 4Cos(\theta))$$

$$\psi_1 = (-1 + r^2)^2 * (6 + r * Cos(\theta))$$

$$T_1 = (-(1/96)) * r * (-1 + r^2) * (195 - 99 * r^2 + r^4 - 4 * r * (-5 + 3 * r^2) * Cos(\theta)) * Sin(\theta)$$

$$\psi_2 = (\frac{1}{240}) * r * (-1 + r^2)^2 * (r * (-1073 + 194 * r^2 + r^4) * Cos[o] + 2 * (55 - 58 * r^2 + 17 * r^4 + 4 * r^2 * (-4 + r^2) * Cos(2\theta))) * Sin(\theta)$$

$$T_2 = -((\frac{2}{806400}) * ((-1 + r^2) * (67375 - 249725 * r^2 + 334075 * r^4 - 188825 * r^6 + 37450 * r^8 - 350 * r^{10} + 56 * r * (-36994 + 42746 * r^2 - 21429 * r^4 + 3921 * r^6 + 21 * r^8) * Cos(\theta) + r^2 * (-133197 + 195943 * r^2 - 114962 * r^4 + 24898 * r^6 + 48 * r^8) * Cos(2\theta) + (133864 * r^3 - 104276 * r^5 + 10300 * r^7 + 780 * r^9)Cos(3\theta) + (3168 * r^4 - 2880 * r^6 + 480 * r^8)Cos(4\theta))))$$

$$\psi_3 = -((\frac{1}{84672000}) * ((-1 + r^2)^2 * (-30 * r * (-2456781 + 2641279 * r^2 - 1702779 * r^4 + 591683 * r^6 - 82422 * r^8 + 33 * r^{10}) * Cos(\theta) + 7 * (60 * r^2 * (-893734 + 573248 * r^2 - 199193 * r^4 + 21246 * r^6 + 171 * r^8) * Cos(2\theta) + r^3 * (-2134693 + 2251150 * r^2 - 1008705 * r^4 + 124220 * r^6 + 145 * r^8) * Cos(3\theta) + 60 * (1173471 - 813698 * r^2 + 296129 * r^4 - 66424 * r^6 + 5548 * r^8 + 324 * r^{10} + r^4 * (13437 - 4361 * r^2 - 25 * r^4 + 55 * r^6) * Cos(4\theta) + 2 * r^5 * (111 - 50 * r^2 + 5 * r^4) * (5\theta))))))$$

$$T_3 = -((\frac{1}{13547520000}) * (r * (-1 + r^2) * (25 * (-2761906411 + 3619925021 * r^2 - 2746742895 * r^4 + 1317425521 * r^6 - 345795983 * r^8 + 37564337 * r^{10} - 558223 * r^1 + 10553 * r^{14}) * Sin(\theta) + r * (40 * (-69644431 + 140680628 * r^2 - 144949486 * r^4 + 83221450 * r^6 -$$

$$24036365 * r^{8} + 2727715 * r^{10} + 8215 * r^{12}) * Sin(2\theta) + r * ((7030363853 - 9396790912 * r^{2} + 4833665720 * r^{4} - 1109175800 * r^{6} + 47626000 * r^{8} + 7874050 * r^{10} - 1650 * r^{12}) * Sin(3\theta) + 2 * r * (7 * (25111009 - 44599295 * r^{2} + 30147665 * r^{4} - 8588735 * r^{6} + 738790 * r^{8} + 12190 * r^{10}) * Sin(4\theta) + 200 * r * ((-99845 + 91857 * r^{2} - 8883 * r^{4} - 4746 * r^{6} + 714 * r^{8}) * Sin(5\theta) + 2 * r * (-757 + 875 * r^{2} - 259 * r^{4} + 21 * r^{6}) * Sin(6\theta))))))))$$

$$\psi_4 = -\frac{1}{312947712000000}((r * (-1 + r^{2})^{2} * (-3080 * (-6609492440 + 9671158607 * r^{2} - 8344153866 * r^{4} + 4964414431 * r^{6} - 1984392760 * r^{8} + 437992845 * r^{10} - 37905470 * r^{12} + 165515 * r^{14}) * Sin(\theta) + r * (70 * (-3972648131629 + 3086685518956 * r^{2} - 1686494511546 * r^{4} + 556724504840 * r^{6} - 104309018270 * r^{8} + 7006783620 * r^{10} + 84882080 * r^{12} + 356140 * r^{14}) * Sin(2\theta) + r * (56 * (-160064781199 + 256765755340 * r^{2} - 198151227225 * r^{4} + 79696231850 * r^{6} - 17235631025 * r^{8} + 1348918650 * r^{10} + 5645025 * r^{12}) * Sin(3\theta) + 5 * r * (7 * (238767947023 - 218964205418 * r^{2} + 94120066585 * r^{4} - 13203737060 * r^{6} - 295654865 * r^{8} + 127268490 * r^{10} + 29925 * r^{12}) * Sin(4\theta) + 8 * r * (3 * (2820035424 - 3867980543 * r^{2} + 2130373098 * r^{4} - 416502646 * r^{6} + 22277360 * r^{8} + 711480 * r^{10}) * Sin(5\theta) + 70 * r * ((-6098261 + 1872428 * r^{2} + 769272 * r^{4} - 299124 * r^{6} + 26334 * r^{8}) * Sin(6\theta) + 24 * r * (-3112 + 1995 * r^{2} - 378 * r^{4} + 21 * r^{6}) * Sin(7\theta)))))))))$$

The global variables considered here are the Nusselt numbers represent the rate of heat transfer introduced by:

$$Nu = \frac{1}{\pi}\int_0^{\pi}\frac{\partial T}{\partial r}d\theta, \qquad Nu1 = \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\partial T}{\partial r}d\theta,$$

$$Nu = -\frac{1}{\pi}\frac{\lambda}{384}\left(\frac{97}{24} - \frac{2392493017}{376320000}\left(\frac{\lambda}{384}\right)^{2} - \frac{48864363455829390255707299 3}{2188263542778593280000000 00}\left(\frac{\lambda}{384}\right)^{4} + \frac{826811856022283504771786850 7083}{875972108070803856752640000 00000}\left(\frac{\lambda}{384}\right)^{6} + \cdots\right.$$

$$(Nu1 = (-2\pi + \frac{418631}{5600}\left(\frac{\lambda}{384}\right)^{2} - \frac{1373243788695305 81}{142391208960000 00}\left(\frac{\lambda}{384}\right)^{4} + \frac{32757081766082871494641795966765068538918777153 4159}{33108992043432783808049008193977712640000000000 000}\left(\frac{\lambda}{384}\right)^{6} + \cdots$$

ANALYSIS OF SERIES AND DISCUSSION

Pade approximants has been used in original forms to enable us to increase the range of applicability of the series as has been used in the works of Mansour [2] and Mansour [3]. This method does not necessarily require any information about the radius of convergence. The Pade approximants provide an approximation that is invariant under an Euler transformation of the independent variables. The theory of Pade approximants has been used extensively in Mansour [8]. Briefly stated, the Pade approximant is the ratio P (λ)/Q (λ) of polynomials P and Q of degree m and n, respectively, that, when expanded, agrees with the given series through terms of degree m+n, and normalized by P (Q) =1. Such rational fractions are known to have remarkable properties of analytic continuation. The coefficients of the power series must be known to degree m+n. By equating like power of g(λ) and P(λ)/Q(λ), the linear system of m+n+1 equation must be solved to obtain the coefficients in the functional form P(λ)/Q(λ) Pade approximation of orders [1/2], [2/3] and [3/4] for Nu1 series, are respectively:

pade[1/2]:

$$\left(\frac{2 + \frac{\pi}{2}}{1 - \frac{18631\lambda^{2}}{2800(4+\pi)}}\right)$$

pade[2/3]:

$$\left(\frac{2+\frac{\pi}{2}+\lambda^2\left(\frac{18631}{5600}+\frac{137324378869530581\left(2+\frac{\pi}{2}\right)}{47373046680960000}\right)}{1+\frac{137324378869530581\lambda^2}{47373046680960000}}\right)$$

When we form the ratios [8/9], [7/8], [6/7], [5/6], [4/5], [3/4] and [2/3] of the Pade approximants, It can be shown, they agreed up to the value $K \cong .3$. This conclusion is confirmed as is plotted in Fig (2).

Of course for lack of space we omit showing the calculation further than this order if any reader interested to have more calculation please contacts the author.

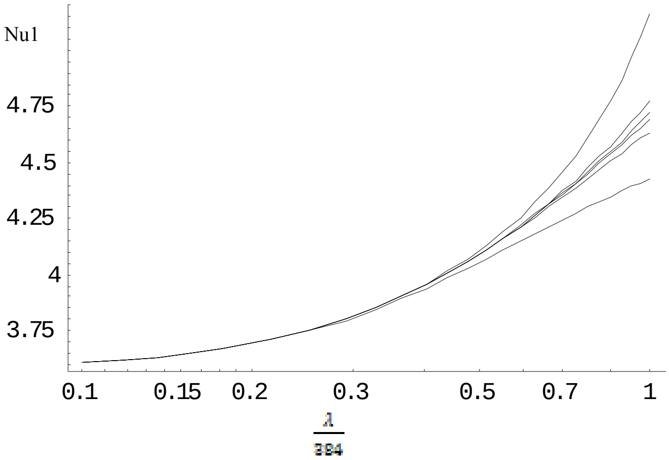

Fig. 2 Plots of [8/9], [7/8], [6/7],[5/6], [4/5], [3/4]and [2/3] of the Pade approximants for Nusselt number series (Nu1) versus λ.

CONCLUSION

We consider the two-dimensional problem of steady natural convection in a Horizontal Cylinder with constant volumetric heat flux filled with viscous fluid subject to cosine temperature variation on the boundary. The solution is expanded in powers of Rayleigh number and the series extended by means of symbolic calculation up to 16 terms. Analysis of these expansions allows the exact computation for arbitrarily accuracy up to 50000 figures. Although the range of exactness is almost the same order as of the radius of convergence but Pade approximation lead our result to be good even for much higher value of the similarity parameterr we have found 16 terms exactly by means of symbolic calculation up to 16th order. Then we tried to make analytic continuation by using Pade approximation. In other words, we have solved the nonlinear partial differential equation exactly by means of computer and that is a real success.

The numerical and experimental results for the same problem but without constant volumetric heat flux were based on the value for water of the Prandtl number 6.45. Our present work assumes that the Prandtl number is unity. As an aside, we note that as long as the magnitude of the Prandtl number is order of unity, there is a small correlation between the value of the Prandtl number and the results of this problem. Numerical works of [10] is in qualitative agreement with our Nusselt number of course it is not the same problem as ours. We repeat our series has finite radius of convergence but by using pade approximant we are able to go beyond that value as far as K=.3 in continuous function of Rayleigh number and that is real success. This method has been used in [12] for analogues problem of steady natural convection in a narrow Horizontal cylindrical annulus successfully. The only qualitative comparable data for the case of circular cavity without constant volumetric heat flux is the experiment done by [13], but their work is for high Prandtl number, but amazingly they are very consistent to our work. Also work of [14] for high Prandtl number with a slightly different wall temperature distribution (saw-tooth rather than cosine) are someway close to our results. Moreover, the experimental work of [15] for the same configuration as ours of course without constant volumetric heat flux and Prandtl number of order unity with different boundry condition, with the cylindrical wall divided along a vertical plane and the two halves maintained at different uniform temperature (step function) is also qualitatively consistent to ours. It is particularly gratifying is our ability to extract from the perturbation series for small gap the behavior for finite gap, of course the gap even as large as the inner radius.,

I t is worth to mention that pade [1/2] and [2/3] are for Prandtl number unity the approximation of high Prandtl number has proven not to be very restrictive. However, it would not be acceptable for a liquid metal like mercury, with Pr= 0.02.

Author in[16] has used similar method as ours for his rectangular geometry and compared other methods with his calculation. But we have not found any literature for our circular cavity with the same boundary conditions, in order to compare with. As we have mentioned several times our calculations are exact that is the point any other methods should be checked with ours.

NOMENCLATURE

R	Radius
β	The coefficient of thermal expansion
g	Gravity
ν	Kinematics viscosity
ψ	Stream function
T	Temperature
ΔT_0	Temperature gradient related to heat flux

$G = \frac{\beta g \Delta T_0 R^3}{v^2}$ Grashof number

$Pr = \frac{v}{\kappa}$ Prandtl number

$Ra = \frac{\beta g \Delta T_0 R^3}{kv} = GPr$ Rayleigh number

$\alpha = \frac{\kappa}{\rho c_p}$ Thermal diffusivity

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